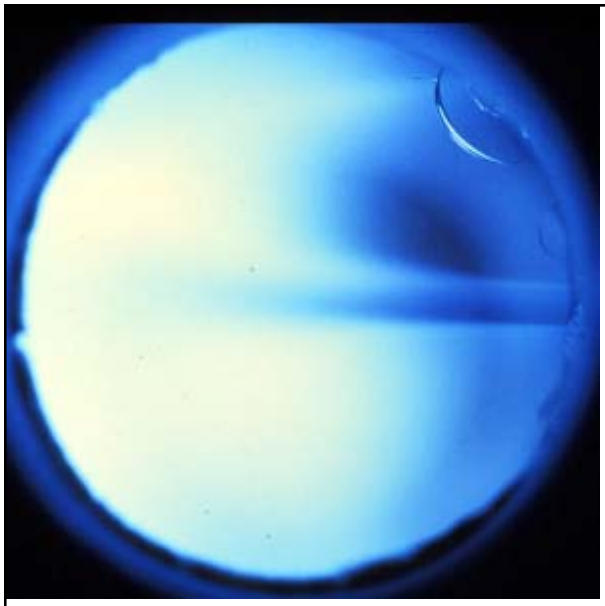

Combustion dynamics
Lecture 2a

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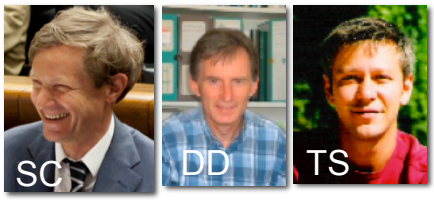


Princeton summer school, June 2019

Acoustics of reactive flows

Accounting for heat rease fluctuations
Compact flames
Acoustic energy balance
Equations of reactive flows

Acoustics of reactive flows

Consider the set of acoustic equations but this time including a nonsteady heat release source term in the energy balance

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = 0$$

$$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} + \boldsymbol{\nabla} p_1 = 0$$

$$\rho_0 T_0 (\frac{\partial}{\partial t} s_1) = \dot{q}_1$$

The state equation $p = p(\rho, s)$ may be differentiated

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s d\rho + \left(\frac{\partial p}{\partial s}\right)_\rho ds$$

$$p_1 = \left(\frac{\partial p}{\partial \rho}\right)_s \rho_1 + \left(\frac{\partial p}{\partial s}\right)_\rho s_1$$

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \quad \alpha = \left(\frac{\partial p}{\partial s}\right)_\rho$$

$$p_1 = c^2 \rho_1 + \alpha s_1$$

For a perfect gas $p = \rho^\gamma \exp(s/c_v)$

$$c^2 = \frac{\gamma p}{\rho} = \gamma r T \qquad \alpha = \frac{p}{c_v} = (\gamma - 1)\rho T$$

The density perturbation may be expressed in terms of pressure and entropy perturbations

$$\rho_1 = \frac{1}{c^2} p_1 - \frac{\alpha}{c^2} s_1$$

This relation may be introduced in the balance of mass

$$\frac{1}{c^2}\frac{\partial p_1}{\partial t} - \frac{\alpha}{c^2}\frac{\partial s_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = 0$$

Using the perturbed energy equation

$$\rho_0 T_0 \left(\frac{\partial}{\partial t} s_1\right) = \dot{q}_1$$

One obtains

$$\frac{1}{c^2}\frac{\partial p_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = \frac{\alpha}{\rho_0 T_0}\frac{1}{c^2}\dot{q}_1$$

The previous equation may be combined with the perturbed momentum balance

$$\frac{\partial}{\partial t} \quad \frac{1}{c^2}\frac{\partial p_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = \frac{\alpha}{\rho_0 T_0}\frac{1}{c^2}\dot{q}_1$$

$$-\boldsymbol{\nabla}\cdot \quad \rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} + \boldsymbol{\nabla} p_1 = 0$$

$$\frac{1}{c^2}\frac{\partial^2 p_1}{\partial t^2} - \boldsymbol{\nabla}^2 p_1 = \frac{\alpha}{\rho_0 T_0}\frac{1}{c^2}\frac{\partial \dot{q}_1}{\partial t}$$

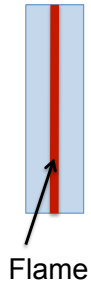
$$\frac{\alpha}{\rho_0 T_0} = \frac{p_0}{\rho_0 c_v T_0} = \gamma - 1$$

One obtains a wave equation with a source term

$$\boxed{\frac{1}{c^2}\frac{\partial^2 p_1}{\partial t^2} - \boldsymbol{\nabla}^2 p_1 = \frac{\gamma - 1}{c^2}\frac{\partial \dot{q}_1}{\partial t}}$$

The compact flame case

$$\frac{1}{c^2}\frac{\partial p_1}{\partial t} + \rho_0 \boldsymbol{\nabla}\cdot\boldsymbol{v}_1 = \frac{\alpha}{\rho_0 T_0}\frac{1}{c^2}\dot{q}_1$$



Now $$\frac{\alpha}{\rho_0 T_0} = \frac{p_0}{\rho_0 c_v T_0} = \gamma - 1$$

Thus $$\frac{1}{c^2}\frac{\partial p_1}{\partial t} + \rho_0 \boldsymbol{\nabla}\cdot\boldsymbol{v}_1 = \frac{\gamma-1}{c^2}\dot{q}_1$$

Or equivalently $$\frac{1}{\rho_0 c^2}\frac{\partial p_1}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{v}_1 = \frac{\gamma-1}{\rho_0 c^2}\dot{q}_1$$

Now $\rho_0 c^2 = \gamma p$ is essentially constant across the flame

Integrating the last expression on a volume including the flame

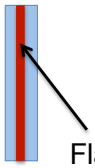
$$\int_V \frac{1}{\rho_0 c^2}\frac{\partial p_1}{\partial t} dV + \int_V \boldsymbol{\nabla}\cdot\boldsymbol{v}_1 dV = \int_V \frac{\gamma-1}{\rho_0 c^2}\dot{q}_1 dV$$

Or equivalently

$$\frac{1}{\rho_0 c^2}\int_V \frac{\partial p_1}{\partial t} dV + \int_V \boldsymbol{\nabla}\cdot\boldsymbol{v}_1 dV = \frac{\gamma-1}{\rho_0 c^2}\int_V \dot{q}_1 dV$$

If the flame is compact, the first term vanishes. The second term may be transformed using Green's theorem yielding

$$S_2 v_2' - S_1 v_1' = \frac{\gamma-1}{\rho_0 c^2}\dot{Q}' \quad \text{where} \quad \dot{Q}' = \int_V \dot{q}' dV$$



Assume that the surfaces on the upstream and downstream sides are equal

$$v_2' - v_1' = \frac{\gamma - 1}{\rho_0 c^2} \frac{1}{S} \dot{Q}'$$

From the definition of the flame transfer function

$$\mathcal{F}(\omega) = \frac{\dot{Q}'/\overline{\dot{Q}}}{v'/\overline{v}}$$

$$\dot{Q}' = \overline{\dot{Q}} \mathcal{F}(\omega) v'/\overline{v}$$

$$v_2' - v_1' = \frac{\gamma - 1}{\rho_0 c^2} \frac{\overline{\dot{Q}}}{S\overline{v}} \mathcal{F}(\omega) v_1'$$

Now $$\overline{\dot{Q}} = \dot{m} c_p (T_b - T_u)$$

and $$\rho_0 c^2 S\overline{v} = \dot{m} \gamma r T_u$$

$$\frac{\gamma - 1}{\rho_0 c^2} \frac{\overline{\dot{Q}}}{S\overline{v}} = \frac{\gamma - 1}{\gamma r} c_p \frac{T_b - T_u}{T_u} = \frac{T_b}{T_u} - 1$$

Hence

$$\boxed{v_2' - v_1' = \left(\frac{T_b}{T_u} - 1\right) \mathcal{F}(\omega) v_1'}$$

Acoustic energy balance

$$p_1 \quad \frac{1}{\rho_0 c^2}\frac{\partial p_1}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{v}_1 = \frac{\gamma-1}{\rho_0 c^2}\dot{q}_1$$

$$\boldsymbol{v}_1\cdot \quad \rho_0\frac{\partial \boldsymbol{v}_1}{\partial t} + \boldsymbol{\nabla} p_1 = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{p_1^2}{\rho_0 c^2} + \frac{1}{2}\rho_0 v_1^2\right) + \boldsymbol{\nabla}\cdot p_1\boldsymbol{v}_1 = \frac{\gamma-1}{\rho_0 c^2}p_1\dot{q}_1$$

$$\mathcal{E} = \frac{1}{2}\frac{p_1^2}{\rho_0 c^2} + \frac{1}{2}\rho_0 v_1^2 \qquad \boldsymbol{\mathcal{F}} = p_1\boldsymbol{v}_1 \qquad \mathcal{S} = \frac{\gamma-1}{\rho_0 c^2}p_1\dot{q}_1$$

Acoustic energy density — Acoustic energy flux — Source term

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = \mathcal{S}}$$

$$\mathcal{S} = \frac{\gamma-1}{\rho_0 c^2}p_1\dot{q}_1$$

Source term

The energy balance should include a term associated with damping processes and takes the final form

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = \mathcal{S} - \mathcal{D}}$$

Taking the average of the energy balance over a period of oscillation one obtains

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F} = S - D$$

$$S = \frac{\gamma - 1}{\rho_0 c^2} \frac{1}{T} \int_T p_1 \dot{q}_1 dt$$

If the source term S is positive it tends to increase the acoustic energy density. However this energy density will grow locally if the source term is greater than the damping term and the acoustic energy flux leaving the local volume

D. Durox, T. Schuller, N. Noiray, A.L. Birbaud and S. Candel (2009) *Combustion and Flame.* **156**,106-119. The Rayleigh criterion and the acoustic energy balance in unconfined self excited oscillating flames.

The energy balance may be integrated over a volume V containing the reactive region :

$$\int_V \frac{\partial E}{\partial t} dV + \int_V \boldsymbol{\nabla} \cdot \boldsymbol{F} dV = \int_V S dV - \int_V D dV$$

Now $\displaystyle\int_V \boldsymbol{\nabla} \cdot \boldsymbol{F} dV = \int_A \boldsymbol{F} \cdot \boldsymbol{n} dA$

So that $\displaystyle\int_V \frac{\partial E}{\partial t} dV = \int_V S dV - \int_V D dV - \int_A \boldsymbol{F} \cdot \boldsymbol{n} dA$

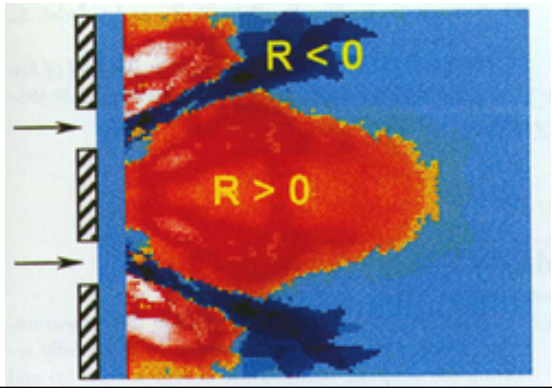
The acoustic energy in the control volume increases if

$$\int_V S dV > \int_V D dV - \int_A \boldsymbol{F} \cdot \boldsymbol{n} dA$$

The Rayleigh criterion

A gain is obtained if pressure and heat-release fluctuations are in phase (Rayleigh, 1878)

$$\frac{1}{T}\int_0^T p'q'dt > 0$$



Local Rayleigh index in a lean premixed Combustor (Lee et al 2000)

driving (R>0) damping (R<0)

Equations of combustion acoustics

$$\frac{d\rho}{dt} = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}$$ Mass

$$\rho\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla}p + \boldsymbol{\nabla}\cdot\boldsymbol{\tau}$$ Momentum

$$\rho c_p\frac{dT}{dt} = \dot{Q} + \frac{dp}{dt} + \boldsymbol{\tau}\!:\!\boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\nabla}\cdot\boldsymbol{J}^H$$ Energy

$$\rho\frac{dY_k}{dt} = \dot{\omega}_k - \boldsymbol{\nabla}\cdot\boldsymbol{J}_k^D$$ Species

$$\boldsymbol{J}_k^D = \rho Y_k \boldsymbol{V}_k^D \qquad \boldsymbol{J}^H = -\lambda\boldsymbol{\nabla}T + \sum_{k=1}^{N}\rho Y_k \boldsymbol{V}_k^D h_k$$ Diffusion and heat fluxes

$$\dot{Q} = -\sum_{k=1}^{N} h_k\dot{\omega}_k$$ Heat release rate

Starting from the state equation for the mixture

$$p = \rho r_g T, \text{ where } r_g = R/W \qquad \frac{1}{W} = \sum_{k=1}^{N} \frac{Y_k}{W_k}$$

one obtains

$$dT = \left(\frac{\partial T}{\partial p}\right)_{\rho, Y_k} dp + \left(\frac{\partial T}{\partial \rho}\right)_{p, Y_k} d\rho + \left(\frac{\partial T}{\partial Y_k}\right)_{\rho, p} dY_k$$

$$dT = \frac{T}{p} dp - \frac{T}{\rho} d\rho - T \sum_{k=1}^{N} \frac{W}{W_k} dY_k$$

- Combining the previous expression with the balance equations for energy and species one obtains

$$\frac{1}{\gamma p}\frac{dp}{dt} - \frac{1}{\rho}\frac{d\rho}{dt} = \frac{\dot{Q}}{\rho c_p T} + W \frac{d}{dt}\left(\frac{1}{W}\right)$$
$$+ \frac{1}{\rho c_p T}\left[\boldsymbol{\nabla} \cdot \lambda \boldsymbol{\nabla} T + \boldsymbol{\tau} : \boldsymbol{\nabla} \boldsymbol{v} - \sum_{k=1}^{N} \rho Y_k c_{pk} \boldsymbol{V}_k^D \cdot \boldsymbol{\nabla} T\right]$$

- Together with the balance of mass and momentum,

$$\frac{d\rho}{dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{v} \quad \text{and} \quad \rho \frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}$$

this expression yields a wave equation for the logarithm of the pressure

Wave equation in a reactive flow

$$\nabla \cdot \left(\frac{c^2}{\gamma} \nabla \ln p \right) - \frac{d}{dt} \left(\frac{1}{\gamma} \frac{d}{dt} \ln p \right) = \nabla \cdot \left(\frac{1}{\rho} \nabla \cdot \tau \right)$$

$$- \frac{d}{dt} \left(\frac{1}{\rho c_p T} \left[\nabla \cdot \lambda \nabla T + \boldsymbol{\tau} : \nabla \boldsymbol{v} - \sum_{k=1}^{N} \rho Y_k c_{pk} \boldsymbol{V}_k^D \cdot \nabla T \right] \right)$$

$$- \frac{d}{dt} \left(\frac{\dot{Q}}{\rho c_p T} \right) - \frac{d}{dt} \left[W \frac{d}{dt} \left(\frac{1}{W} \right) \right] - \nabla \boldsymbol{v} : \nabla \boldsymbol{v}$$

Combustion noise source associated with nonsteady heat release

Combustion source associated with changes in molar composition

Aerodynamic noise source

Simplified wave equation

$$\boxed{\begin{aligned} &\nabla \cdot \left(\frac{c^2}{\gamma} \nabla \ln p \right) - \frac{d}{dt} \left(\frac{1}{\gamma} \frac{d}{dt} \ln p \right) = \\ &\qquad - \frac{d}{dt} \left(\frac{\dot{Q}}{\rho c_p T} \right) - \frac{d}{dt} \left[W \frac{d}{dt} \left(\frac{1}{W} \right) \right] - \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \end{aligned}}$$

$$\dot{Q} = - \sum_{k=1}^{N} \dot{\omega}_k h_k = \left(-\Delta h_f^0 \right) \dot{\omega}$$ Heat release rate

$\dot{Q}$ $[ML^{-1}T^{-3}]$

$\dot{\omega}$ $[ML^{-3}T^{-1}]$ Reaction rate

$$\frac{d}{dt}\left(\frac{\dot{Q}}{\rho c_p T}\right)$$

Combustion noise source associated with nonsteady heat release

$$\frac{d}{dt}\left[W\frac{d}{dt}\left(\frac{1}{W}\right)\right]$$

Combustion noise source associated with changes in molar composition

Truffaut and Searby (1998) propose an alternative expression for the second source term

$$\frac{d}{dt}\left(\frac{\dot{n}}{n}\right)$$

where n is the molar concentration

$$n = \sum_{k=1}^{N} n_k = \rho \sum_{k=1}^{N} Y_k / W_k$$

Extra term

Using this expression one finds that

$$\frac{\dot{n}}{n} = \frac{1}{n}\frac{dn}{dt} = W\frac{d}{dt}\left(\frac{1}{W}\right) + \frac{1}{\rho}\frac{d\rho}{dt}$$

which contains an extra term

Linearized wave equation

The previous wave equation may be linearized by writing

$$\ln p \simeq \frac{p'}{\bar{p}}$$

and assuming that the mean pressure is essentially constant (combustion is nearly isobaric). One obtains

$$\nabla \cdot \left(\bar{c}^2 \nabla p'\right) - \frac{\partial^2 p'}{\partial t^2} = -\frac{\partial}{\partial t}\left[(\gamma - 1)\dot{Q}'\right] - \gamma \bar{p} \nabla \boldsymbol{v} : \nabla \boldsymbol{v} + \frac{\gamma \bar{p}}{\bar{W}}\frac{\partial^2 W'}{\partial t^2}$$

Nonsteady heat release source term

Aerodynamic sound

Changes in molar composition

An alternative formulation of the wave equation

Low Mach number limit $d/dt \sim \partial/\partial t$

By developing the logarithm of the pressure and using $\gamma p = \rho c^2$ one obtains

$$\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) - \frac{\partial}{\partial t}\left(\frac{1}{\rho c^2}\frac{\partial}{\partial t}p\right) =$$

$$-\frac{\partial}{\partial t}\left(\frac{\dot{Q}}{\rho c_p T}\right) - \frac{\partial}{\partial t}\left[W\frac{\partial}{\partial t}\left(\frac{1}{W}\right)\right] - \nabla \boldsymbol{v} : \nabla \boldsymbol{v}$$

This expression can be rearranged by adding on both sides the left hand side terms where the density and the sound speed are replaced by their uniform ambient values

$$\nabla \cdot \left(\frac{1}{\rho_0}\nabla p\right) - \frac{\partial}{\partial t}\left(\frac{1}{\rho_0 c_0^2}\frac{\partial}{\partial t}p\right)$$

One obtains

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} =$$

$$+\rho_0 \frac{\partial}{\partial t}\left[\left(\frac{1}{\rho c^2} - \frac{1}{\rho_0 c_0^2}\right)\frac{\partial p}{\partial t}\right] \simeq 0 - \rho_0 \nabla \cdot \left[\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\nabla p\right]$$

Indirect noise source

$$-\rho_0 \nabla \mathbf{v} : \nabla \mathbf{v} - \rho_0 \frac{\partial}{\partial t}\left[W\frac{\partial}{\partial t}\left(\frac{1}{W}\right)\right] - \rho_0 \frac{\partial}{\partial t}\left(\frac{\dot{Q}}{\rho c_p T}\right)$$

Aerodynamic noise source

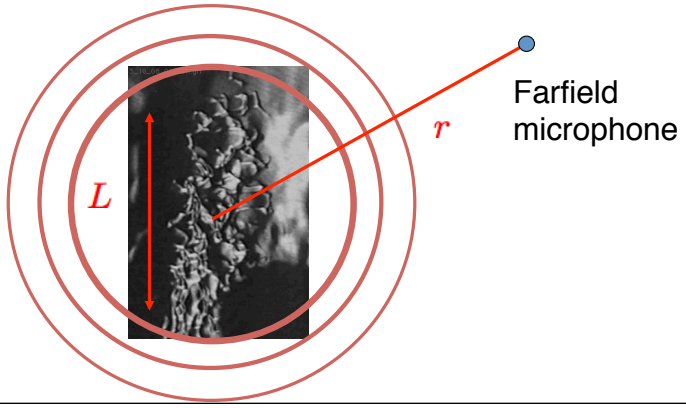
Combustion noise source

The reasoning parallels that used by Lighthill in his theory of aerodynamic sound. This was used by Howe and by Dowling.

Noting that $$\frac{1}{\rho c_p T} = \frac{\gamma - 1}{\rho_0 c_0^2}$$

And only keeping the heat release source term and one finally obtains the following equation

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\gamma - 1}{c_0^2}\frac{\partial \dot{Q}'}{\partial t}$$



- Assuming that radiation takes place in an unconfined domain and only keeping the source term associated with perturbations in heat release one obtains

$$p'(\boldsymbol{r}, t) = \frac{\gamma - 1}{4\pi {c_0}^2}\int_V \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|}\frac{\partial}{\partial t}\dot{Q}'(\boldsymbol{r}_0, t - |\boldsymbol{r} - \boldsymbol{r}_0|/c_0)dV(\boldsymbol{r}_0)$$

- When the observation point is in the farfield of a compact flame the previous expression becomes (Strahle (1985))

$$\boxed{p'(\boldsymbol{r}, t) = \frac{\gamma - 1}{4\pi {c_0}^2 r}\frac{\partial}{\partial t}\int_V \dot{Q}'(\boldsymbol{r}_0, t - r/c_0)dV(\boldsymbol{r}_0)}$$