Flow perturbation is produced
This induces a combustion perturbation
Acoustic feedback links the unsteady combustion process to flow perturbation
The system is unstable if the gain exceeds the damping
A simplified instability model

In region 1 (upstream of the flame)

\[ p_1 = A_1 \exp(ikx) + B_1 \exp(-ikx) \]
\[ v_1 = \frac{1}{\rho_0 c} [A_1 \exp(ikx) - B_1 \exp(-ikx)] \]

In region 2 (downstream of the flame)

\[ p_2 = A_2 \exp(ikx) + B_2 \exp(-ikx) \]
\[ v_2 = \frac{1}{\rho_0 c} [A_2 \exp(ikx) - B_2 \exp(-ikx)] \]
Across the flame, the pressure is continuous, the jump in velocity fluctuations is governed by fluctuations in heat release rate which caused by velocity fluctuations.

\[
\begin{align*}
\rho_1 &= \rho_2 \\
v_2 - v_1 &= \mathcal{F}(\omega)v_1
\end{align*}
\]

In this expression \(\mathcal{F}(\omega)\) designates the flame transfer function multiplied by \((T_b/T_u) - 1\).

The left and right conditions correspond to rigid walls \(v_1(0) = 0\) \(v_2(l) = 0\).

Dispersion relation

\[
\sin(kl) + \mathcal{F}(\omega)\cos(kb)\sin(ka) = 0
\]

It is convenient to define

\[
\mathcal{H}(\omega) = \sin(kl) \\
\mathcal{L}(\omega) = \sin(ka)\cos(kb)
\]

In the absence of a flame, the resonant modes are given by

\[
\mathcal{H}(\omega_0) = 0
\]

The first root corresponds to

\[
\omega_0 = \pi c/l \quad f_0 = c/(2l) \quad \lambda = 2l
\]

Half wave mode
Assuming that the flame response is weak and expanding to first order one obtains
\[ \mathcal{H}(\omega_0) + \left[ \frac{d\mathcal{H}}{d\omega} \right]_{\omega_0} \omega_1 + \mathcal{F}(\omega_0) \mathcal{L}(\omega_0) = 0 \]

Since \( \mathcal{H}(\omega_0) = 0 \)

One obtains the first order estimate
\[ \omega_1 = -\frac{\mathcal{F}(\omega_0) \mathcal{L}(\omega_0)}{\left[ d\mathcal{H}/d\omega \right]_{\omega_0}} \]

\[ \mathcal{L}(\omega_0) = \sin(\pi a/l) \cos(\pi b/l) \]

\[ \left[ d\mathcal{H}/d\omega \right]_{\omega_0} = (l/c) \cos(\omega_0 l/c) = -(l/c) \]

The sign of the imaginary part of the angular frequency defines the stability of this system. If the sign is positive, the system is unstable

\[ \frac{\omega_1 l}{c} = G(\omega) \sin[\phi(\omega_0)] \sin(\pi a/l) \cos(\pi b/l) \]

In general \( b/l > 1/2 \) hence \( \cos(\pi b/l) < 0 \)

and the first mode will be linearly unstable if

\[ \pi < \phi(\omega_0) < 2\pi \quad \text{modulo} \quad 2\pi \]
One may now consider a situation where the duct is closed on the upstream side and open downstream.

\[ \begin{array}{c}
\rightarrow A_1 \\
\leftarrow B_1 \\
\rightarrow A_2 \\
\leftarrow B_2 \\
\hline
a \\
\hline
l \\
b
\end{array} \]

This case may be worked out as before. One has to change the boundary condition at \( x=l \).

The dispersion relation becomes

\[ \mathcal{H}(\omega) = \cos kl - \mathcal{F}(\omega) \sin ka \sin kb = 0 \]

In the absence of a flame the dispersion relation becomes

\[ \cos kl = 0 \]

The first root of this expression is given by

\[ \omega_0 = \frac{\pi c}{2l} \quad \text{corresponding to} \quad f_0 = \frac{c}{4l} \quad \lambda = 4l \]

Quarter wave mode

Assuming that the flame response is weak and expanding to first order one obtains

\[ \mathcal{H}(\omega_0) + \left[ \frac{d\mathcal{H}}{d\omega} \right]_{\omega_0} \omega_1 + \mathcal{F}(\omega_0)\mathcal{L}(\omega_0) = 0 \]

where

\[ \mathcal{L}(\omega) = -\sin ka \sin kb \]
Since $\mathcal{H}(\omega_0) = 0$

One obtains the first order estimate

$$\omega_1 = -\frac{\mathcal{F}(\omega_0) \mathcal{L}(\omega_0)}{[d\mathcal{H}/d\omega]_{\omega_0}}$$

This yields after some calculations

$$\omega_1 = -\frac{c}{l} \mathcal{F}(\omega_0) \sin k_0 a \sin k_0 b$$

The imaginary part of the angular frequency perturbation is given by

$$\omega_{1i} = -\frac{c}{l} \mathcal{G}(\omega_0) \sin \phi \sin \frac{\pi a}{2l} \sin \frac{\pi b}{2l}$$

The system is unstable if $\pi < \phi < 2\pi$ modulo $2\pi$