Lecture 4

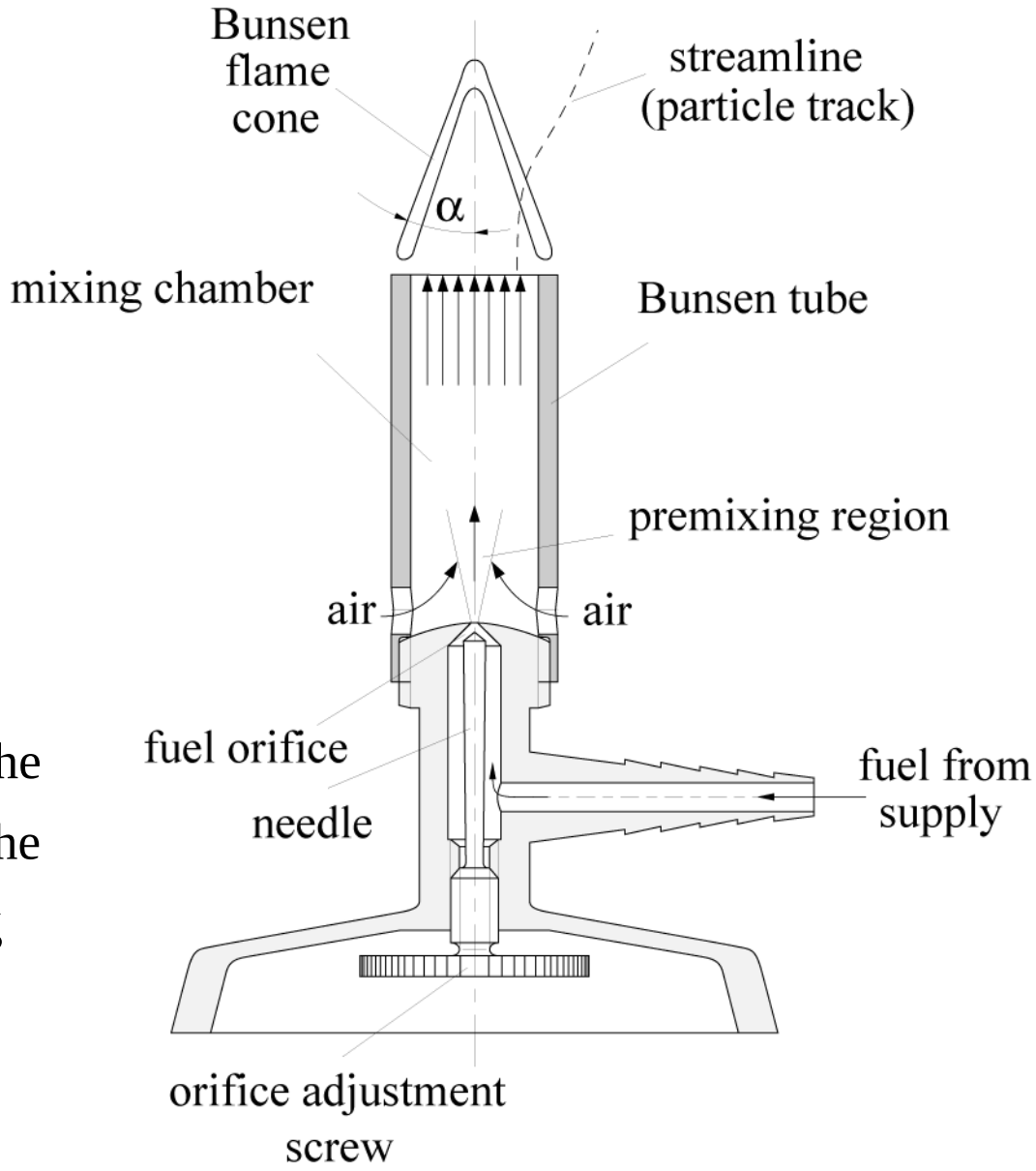
Laminar Premixed Flame Configuration

Bunsen burner

classical device to generate a laminar premixed flame.

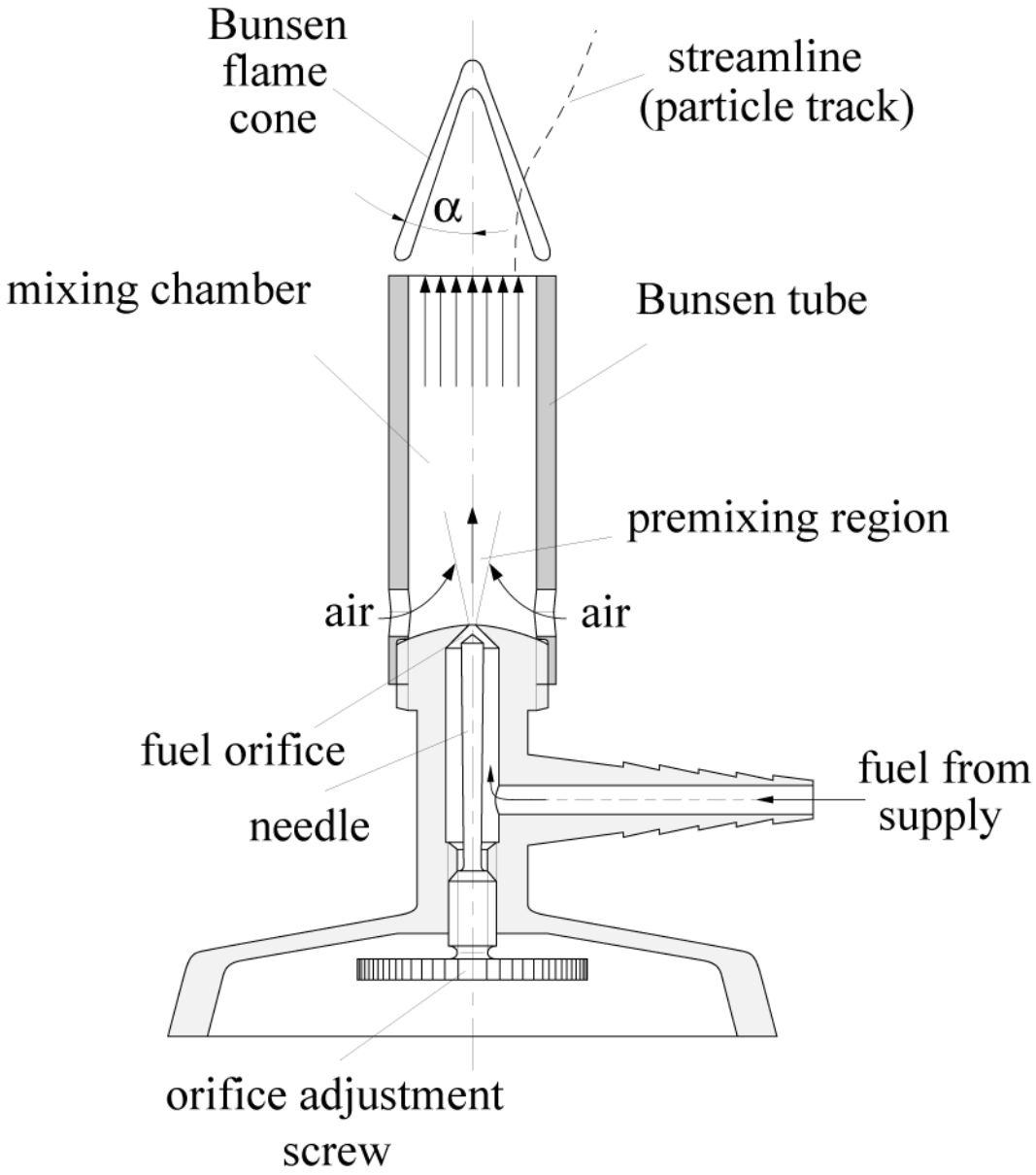
Gaseous fuel enters into the mixing chamber, into which air is entrained.

The velocity of the jet entering into the mixing chamber may be varied and the entrainment of the air and the mixing can be optimized.

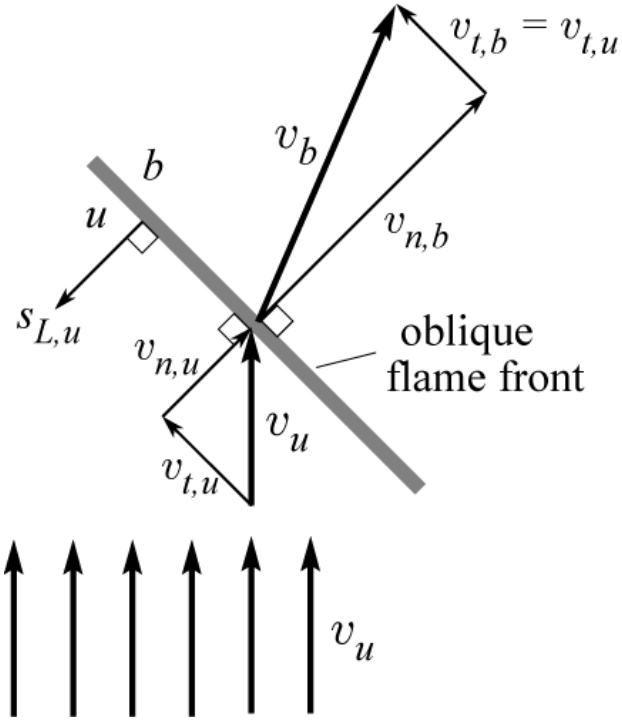


The mixing chamber must be long enough to generate a premixed gas issuing from the Bunsen tube into the surroundings.

If the velocity of the issuing flow is larger than the laminar burning velocity to be defined below, a Bunsen flame cone establishes at the top of the tube.



Kinematic balance for a steady oblique flame:



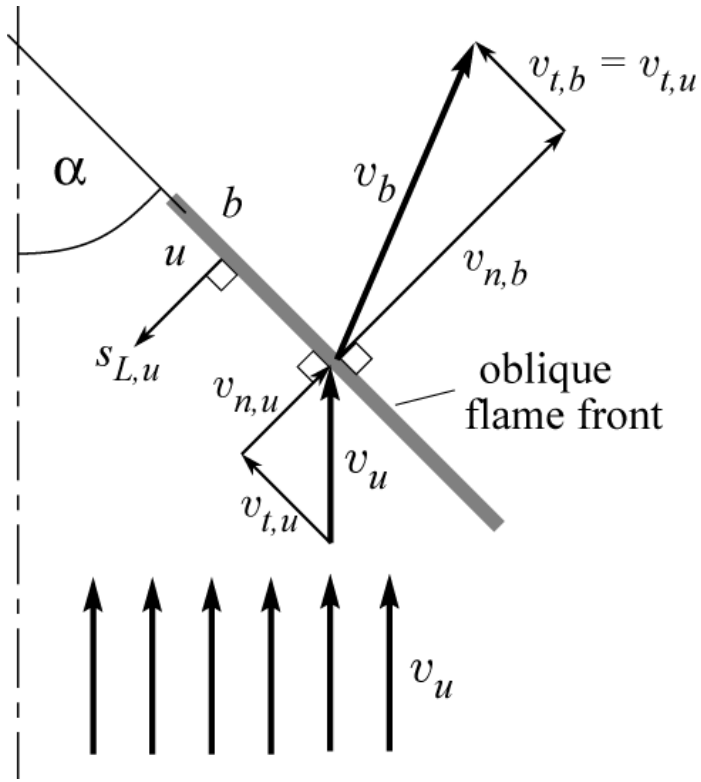
The oncoming flow velocity v_u of the unburnt mixture (subscript *u*) is split into a component $v_{t,u}$ which is tangential to the flame and into a component $v_{n,u}$ normal to the flame front.

Due to thermal expansion within the flame front the normal velocity component is increased, since the mass flow density ρv_n through the flame must be the same in the unburnt mixture and in the burnt gas (subscript *b*).

$$(\rho v_n)_u = (\rho v_n)_b \quad \Rightarrow \quad v_{n,b} = v_{n,u}\frac{\rho_u}{\rho_b}$$

The tangential velocity v_t is not affected by the gas expansion: $v_{t,b} = v_{t,u}$

Vector addition of the velocity components in the burnt gas leads to v_b which points into a direction which is deflected from the flow direction of the unburnt mixture.



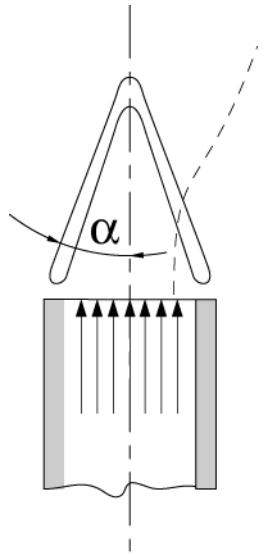
Since the flame front is stationary, the burning velocity $s_{L,u}$ with respect to the unburnt mixture must be equal to the flow velocity of the unburnt mixture normal to the front

$$s_{L,u} = v_{n,u} = v_u \sin \alpha$$

where α is the Bunsen flame cone angle.

This allows to experimentally determine the burning velocity by measuring the cone angle α under the condition that the flow velocity v_u is uniform across the tube exit.

A particular phenomenon occurs at the flame tip.



If the tip is closed, the burning velocity at the tip is equal to the flow velocity

$$s_{L,u}|_{\text{flame tip}} = v_u$$

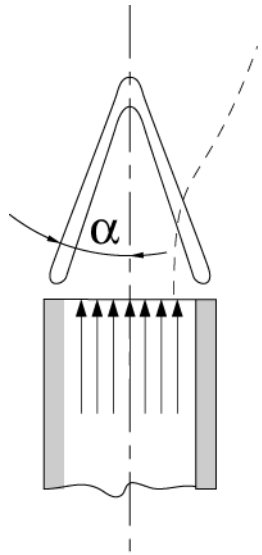
Therefore the burning velocity at the flame tip is by a factor 1 / sin α larger than the burning velocity through the oblique part of the cone.

Explanation: the strong curvature of the flame front at the tip leads to a preheating by the lateral parts of the flame front and thereby to an increase in burning velocity.

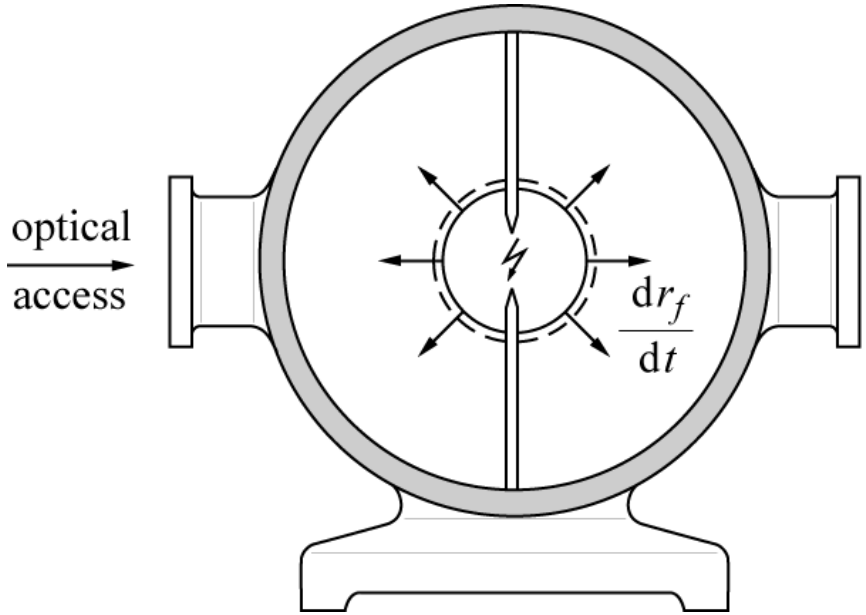
A detailed analysis of this phenomenon also includes the effect of non-unity Lewis numbers by which, for instance, a difference between lean hydrogen and lean hydrocarbon flames can be explained.

Finally, one observes that the flame is detached from the rim of the burner.

This is due to conductive heat loss to the burner which leads in regions very close to the rim to temperatures, at which combustion cannot be sustained.



Another example for an
experimental device to measure
laminar burning velocities
is the combustion bomb
within which a flame
is initiated by a central spark.

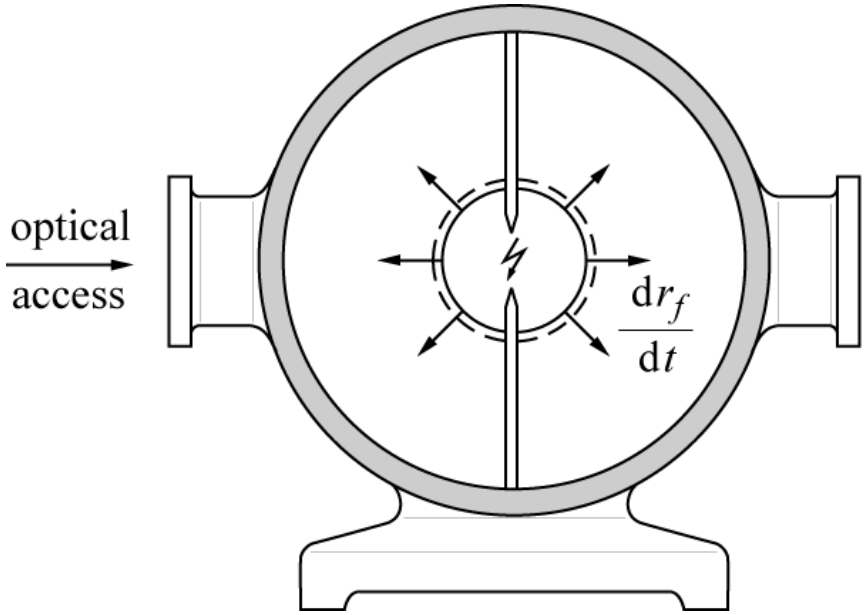


Spherical propagation of a flame then takes place
which may optically be detected through
quartz windows and the flame propagation velocity dr_f/dt may be recorded.

Now the flame front is not stationary.

If the radial flow velocities are defined positive in inward direction, the velocity of the front must be subtracted from these in the mass flow balance through the flame front.

$$\rho u(v_u - \frac{\mathrm{d}r_f}{\mathrm{d}t}) = \rho_b(v_b - \frac{\mathrm{d}r_f}{\mathrm{d}t})$$



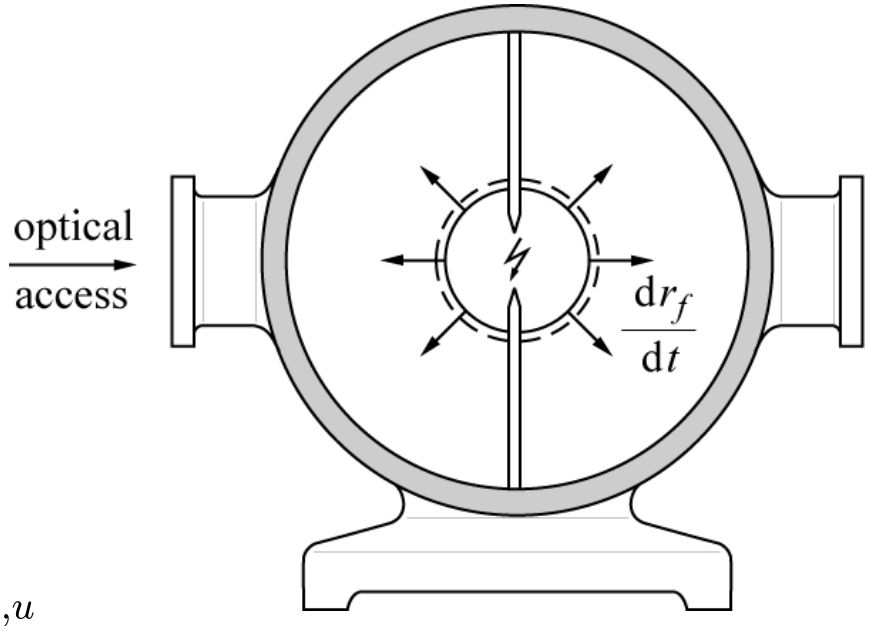
At the flame front the kinematic balance between propagation velocity, flow velocity and burning velocity with respect to the unburnt mixture is

$$\frac{\mathrm{d}r_f}{\mathrm{d}t} = v_u + s_{L,u}$$

Similarly, the kinematic balance with respect to the burnt gas is

$$\frac{\mathrm{d}r_f}{\mathrm{d}t} = v_b + s_{L,b}$$

In the present example the flow velocity in the burnt gas behind the flame is zero due to symmetry.



This leads to

$$\frac{dr_f}{dt} = \frac{\rho u}{\rho_u - \rho_b} v_u = v_u + s_{L,u}$$

from which the velocity in the unburnt mixture is calculated as

$$v_u = \frac{\rho u}{\rho_u - \rho_b} s_{L,u}$$

This velocity is induced by the expansion of the gas behind the flame front.

Furthermore it follows that the flame propagation velocity dr_f/dt is related to the burning velocity $s_{L,u}$ by

$$\frac{dr_f}{dt} = \frac{\rho_u}{\rho_b} s_{L,u}.$$

Measuring the flame propagation velocity then allows to determine $s_{L,u}$.

Furthermore, from $\frac{dr_f}{dt} = v_b + s_{L,b}$ it follows with $v_b = 0$: $\frac{dr_f}{dt} = s_{L,b}$

The comparison shows that the burning velocity with respect to the burnt gas is by a factor ρ_u/ρ_b larger than that with respect to the unburnt gas.

This is equivalent to the Bunsen burner case with: $v_{n,b} = v_{n,u}\frac{\rho_u}{\rho_b}$

Convenience: $s_L \equiv s_{L,u}$

We keep the notation $s_{L,b}$ for the burning velocity with respect to the burnt gas.

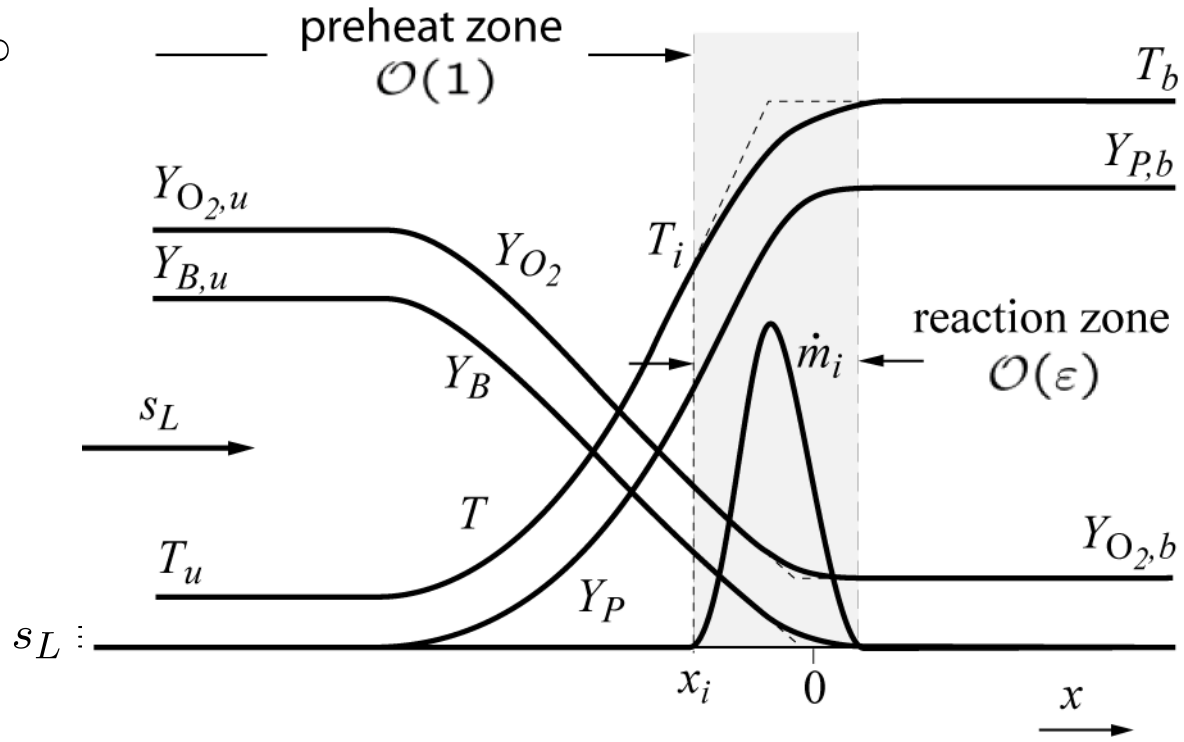
Governing Equations for Steady Premixed Flames,

Numerical Calculations and Experimental Data

Planar steady state flame configuration normal to the *x*-direction

unburnt mixture at $x \rightarrow -\infty$

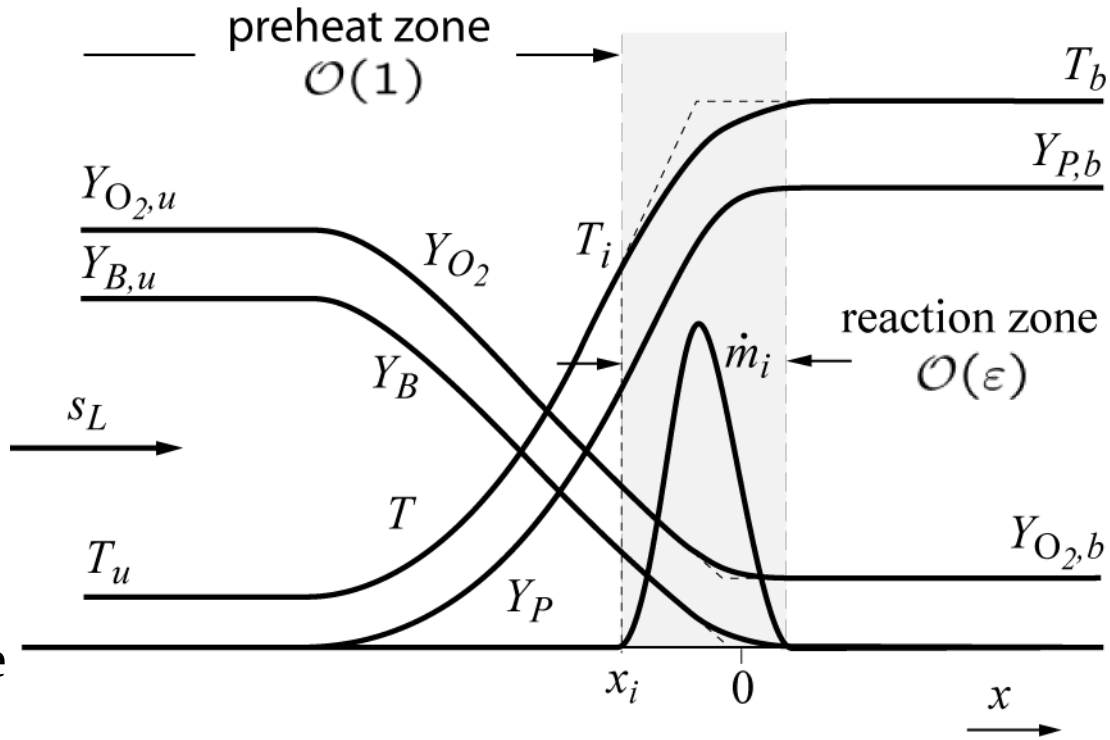
burnt gas at $x \rightarrow +\infty$



The flame structure for the case of a lean flame with a one-step reaction

$$\nu'_F F + \nu'_{O_2} O_2 \rightarrow \nu''_P P$$

The fuel and oxidizer are convected from upstream with the burning velocity s_L
Having the mass fractions $Y_{F,u}$ and $Y_{O_2,u}$ at $x \rightarrow -\infty$ and diffuse into the reaction zone

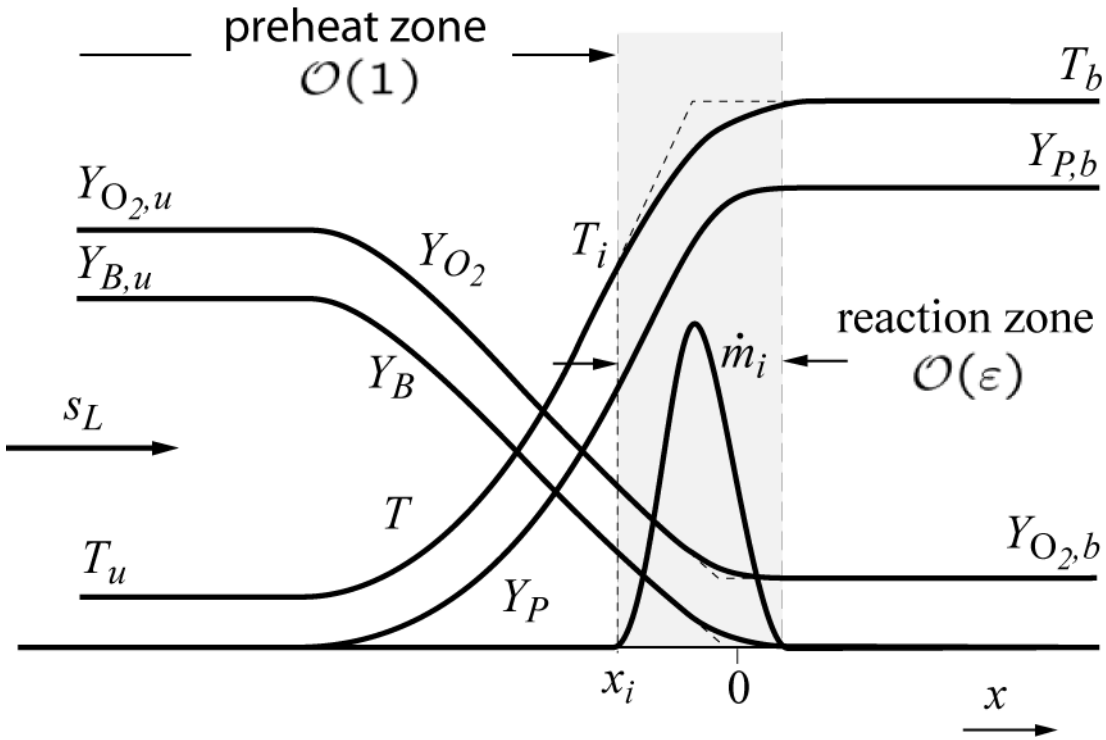


The fuel is entirely depleted while the remaining oxygen is convected downstream where it has the mass fraction $Y_{O_2,u}$.

The chemical reaction forms the product P and releases heat which leads to a temperature rise.

$$\nu'_F F + \nu'_{O_2} O_2 \rightarrow \nu''_P P$$

The mass fraction Y_P increases therefore in a similar way from zero to $Y_{P,b}$ as the temperature from T_u to T_b.



The products diffuses upstream, and mix with the fuel and the oxidizer.

Heat conduction from the reaction zone is also directed upstream leading to a preheating of the fuel/air mixture. Therefore the region upstream of the reaction zone is called the preheat zone.

The fundamental property of a premixed flame, the burning velocity s_L may be calculated by solving the governing conservation equations for the overall mass, species and temperature.

Continuity
$$\frac{\mathrm{d}(\rho u)}{\mathrm{d}x} = 0$$

Species
$$\rho u \frac{\mathrm{d}Y_i}{\mathrm{d}x} = -\frac{\mathrm{d}j_i}{\mathrm{d}x} + \dot{m}_i$$

Energy
$$\rho u c_p \frac{\mathrm{d}T}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda \frac{\mathrm{d}T}{\mathrm{d}x}\right) - \sum_{i=1}^{k} h_i \dot{m}_i - \sum_{i=1}^{k} c_p j_i \frac{\mathrm{d}T}{\mathrm{d}x} + \frac{\partial p}{\partial t}$$

For flame propagation with burning velocities much smaller than the velocity of sound, the pressure is spatially constant and is determined from the thermal equation of state.
Therefore spatial pressure gradients are neglected

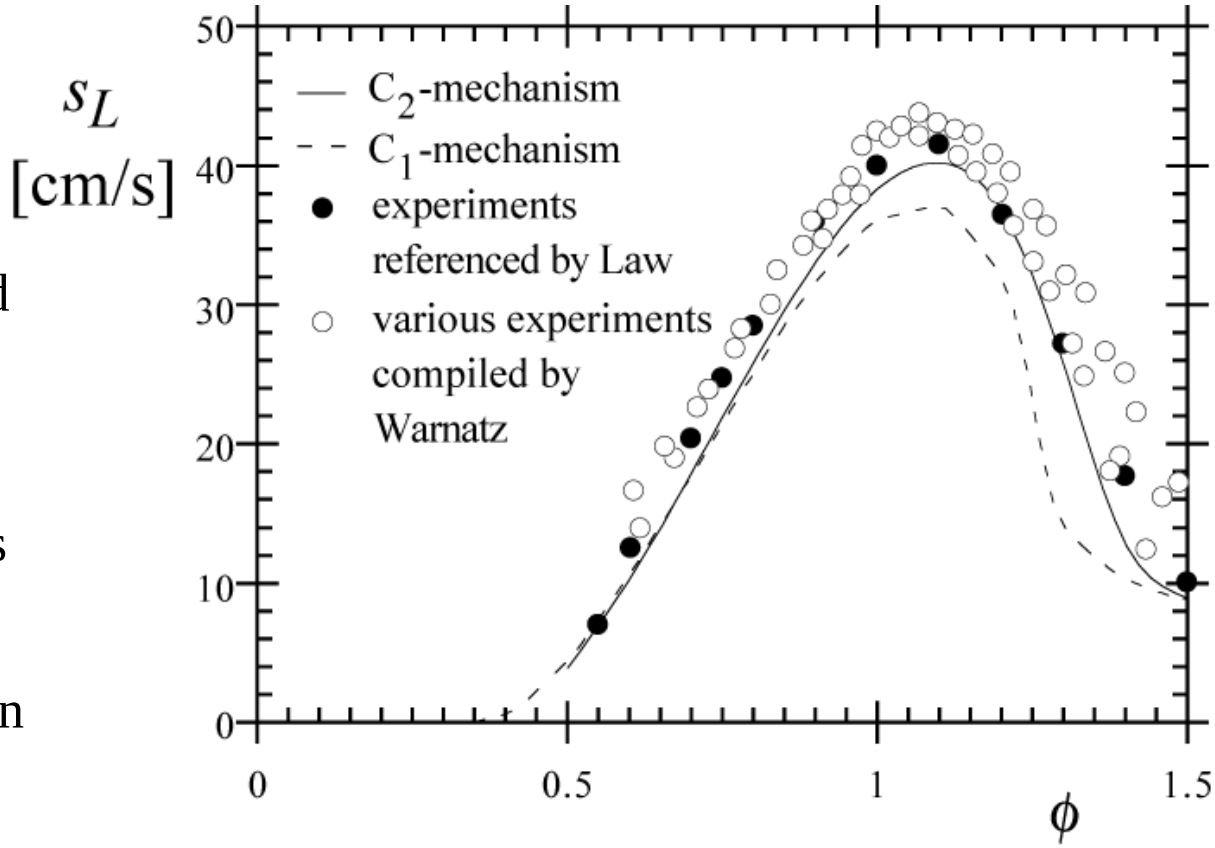
The continuity equation may be integrated once to yield

$$\rho u = \rho_u s_L$$

The burning velocity is an eigenvalue, which must be determined as part of the solution.

The system of equations may be solved numerically with the appropriate upstream boundary conditions for the mass fractions and the temperature and zero gradient boundary conditions downstream.

Example: Calculations of the burning velocity of premixed methane-air flames using a mechanism that contains only C_1-hydrocarbons and a mechanism that includes the C_2-species as a function of the equivalence ratio ϕ [Mauss 1993] .



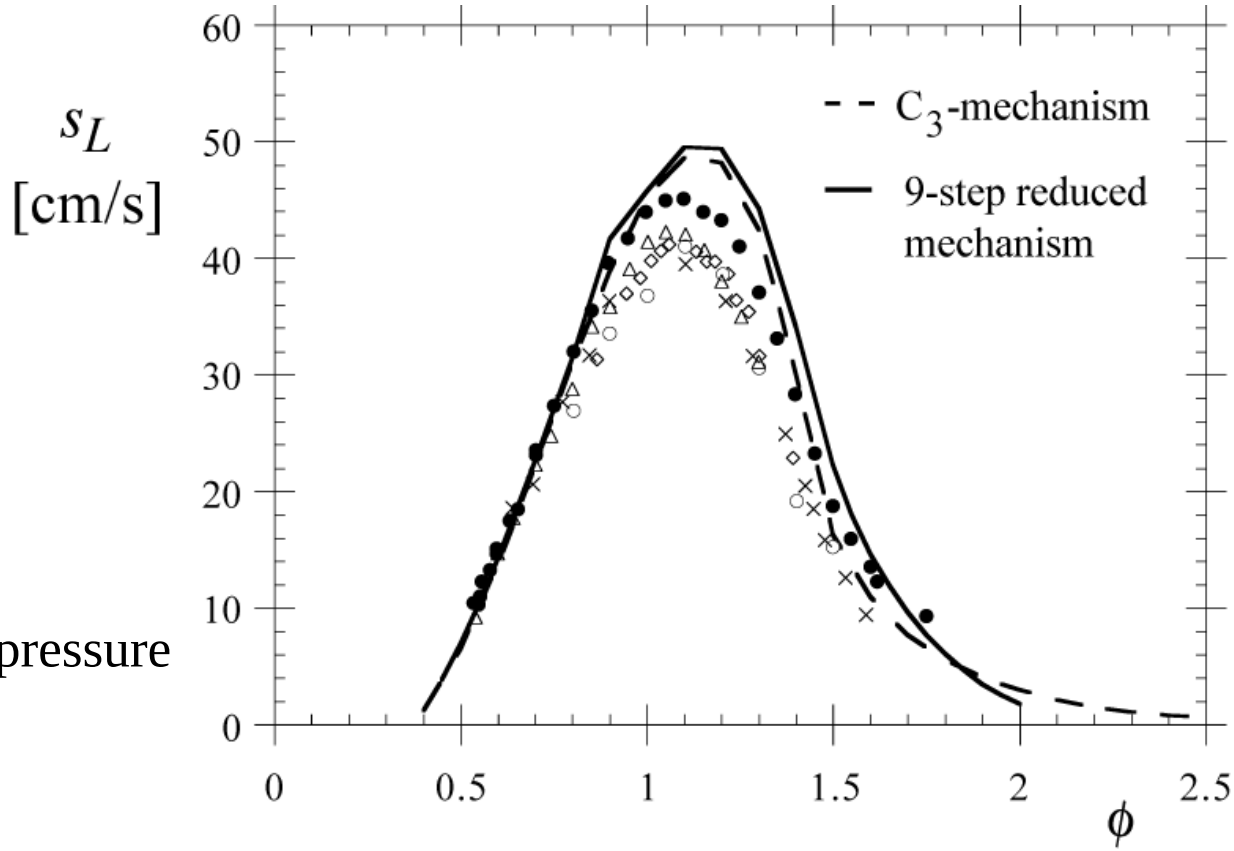
The two curves are compared with compilations of various data from the literature.
It is seen that the calculations with the C_2-mechanism shows a better agreement than the C_1-mechanism.

Example:

burning velocities of propane flames taken from [Kennel1993]

Calculated values of burning velocities for lean flames are compared with approximations given below in Lecture 6 for different pressures.

s_L decreases with increasing pressure but increases with increasing preheat temperature.



The fundamental property of a premixed flame is its ability to propagate normal to itself with a burning velocity.

The burning velocity, to first approximation, depends on thermo-chemical parameters of the premixed gas ahead of the flame only.

In a steady flow of premixed gas a premixed flame will propagate against the flow until it stabilizes.

→ Bunsen flame: the condition of a constant burning velocity is violated at the top of the flame. Curvature must be taken into account.

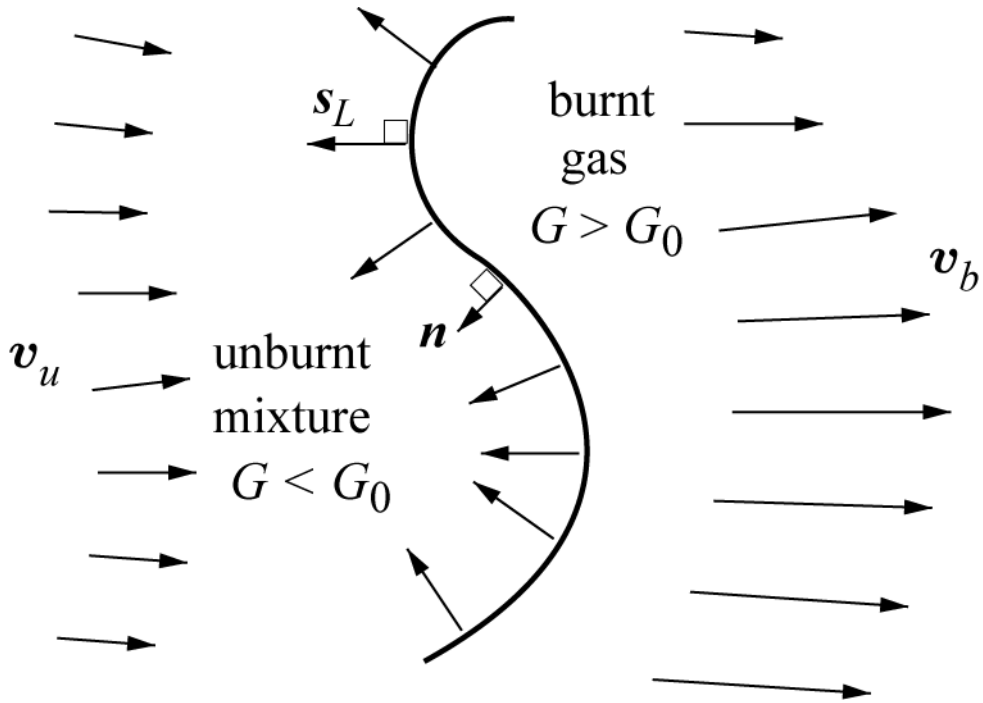
In this chapter we want to calculate flame shapes. We then will consider external influences that locally change the burning velocity and discuss the response of the flame to these disturbances.

A Field Equation Describing the Flame Position

The kinematic relation $\frac{\mathrm{d}r_f}{\mathrm{d}t} = v_u + s_{L,u}$ between the propagation velocity, the flow velocity, and the burning velocity that was derived for spherical flame propagation may be generalized by introducing the vector $\boldsymbol{n}$ normal to the flame

$$\frac{\mathrm{d}\boldsymbol{x}_f}{\mathrm{d}t} = \boldsymbol{v} + s_L\boldsymbol{n},$$

where $\boldsymbol{x}_f$ is the vector describing the flame position,
$\mathrm{d}\boldsymbol{x}_f/\mathrm{d}t$ the flame propagation velocity,
and $\boldsymbol{v}$ the velocity vector.



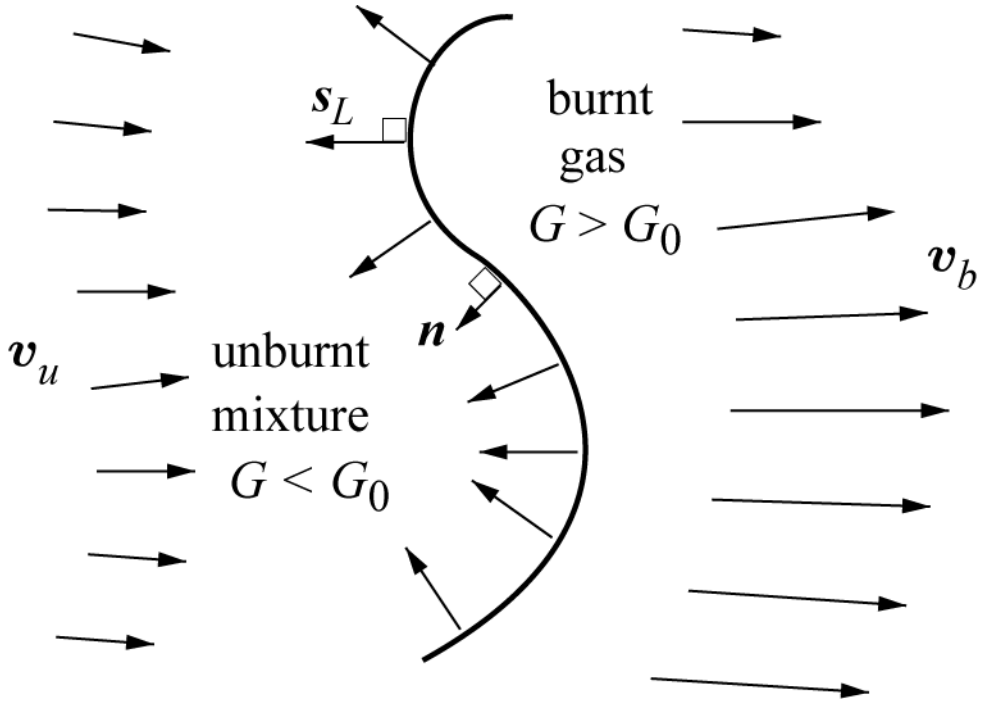
A Field Equation Describing the Flame Position

The normal vector points towards the unburnt mixture and is given by

$$\boldsymbol{n} = -\frac{\boldsymbol{\nabla} G}{|\boldsymbol{\nabla} G|},$$

where $G(\boldsymbol{x},t)$ can be identified as a scalar field whose level surfaces

$$G(\boldsymbol{x}, t) = G_0,$$



where G_0 is arbitrary, represent the flame surface.

The flame contour $G(\boldsymbol{x},t) = G_0$ divides the physical field into two regions, where $G > G_0$ is the region of burnt gas and $G < G_0$ that of the unburnt mixture.

If one differentiates $G(\boldsymbol{x},t) = G_0$ with respect to t at $G = G_0$, such as

$$\frac{\partial G}{\partial t} + \boldsymbol{\nabla} G \cdot \frac{\partial \boldsymbol{x}}{\partial t}\Big|_{G=G_0} = 0,$$

or with $\boldsymbol{n} = -\dfrac{\boldsymbol{\nabla} G}{|\boldsymbol{\nabla} G|}$

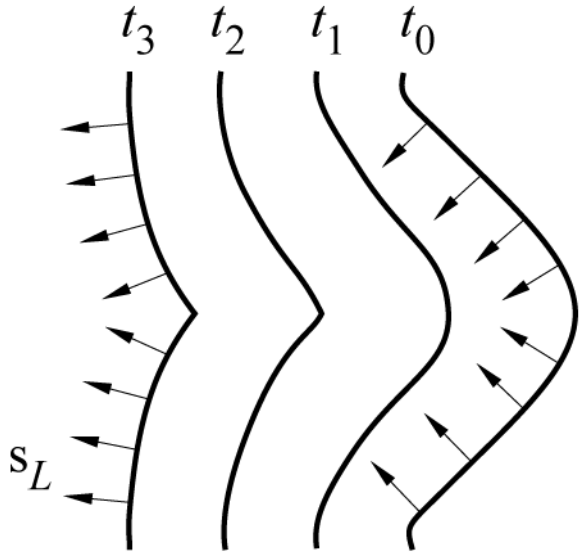
$$\frac{\partial G}{\partial t} - |\boldsymbol{\nabla} G|\, \boldsymbol{n} \cdot \frac{\partial \boldsymbol{x}}{\partial t}\Big|_{G=G_0} = 0$$

Introducing $\dfrac{\mathrm{d}\boldsymbol{x}_f}{\mathrm{d}t} = \boldsymbol{v} + s_L \boldsymbol{n},$ and identifying $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\Big|_{G=G_0} = \dfrac{\mathrm{d}\boldsymbol{x}_f}{\mathrm{d}t}$

one obtains the field equation called G-equation :

$$\frac{\partial G}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} G = s_L |\boldsymbol{\nabla} G|$$

If the burning velocity s_L is defined with respect to the unburnt mixture, then the flow velocity v is defined as the conditioned velocity field in the unburnt mixture ahead of the flame.



For a constant value of s_L the solution of

$$\frac{\partial G}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} G = s_L |\boldsymbol{\nabla} G|$$

is non unique, and cusps will be formed where different parts of the flame intersect.

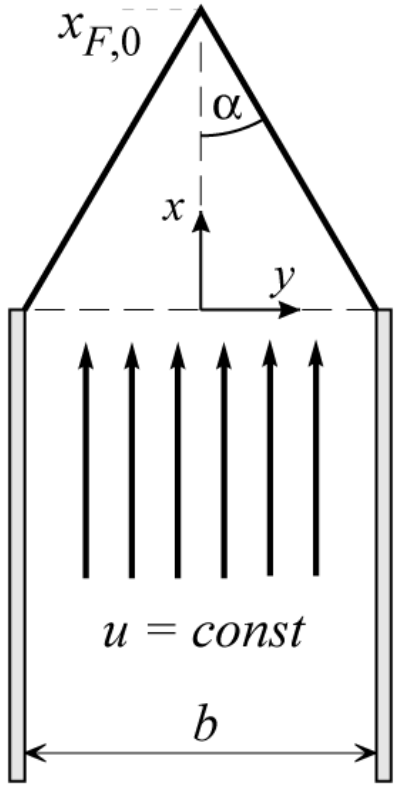
Even an originally smooth undulated front in a quiescent flow will form cusps and eventually become flatter with time. This is called Huygens' principle.

Example

A closed form solution of the G-equation

$$\frac{\partial G}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} G = s_L |\boldsymbol{\nabla} G|$$

can be obtained for the case of a slot burner with a constant exit velocity *u* for premixed combustion,



This is the two-dimensional planar version of the axisymmetric Bunsen burner.

The *G*-equation takes the form

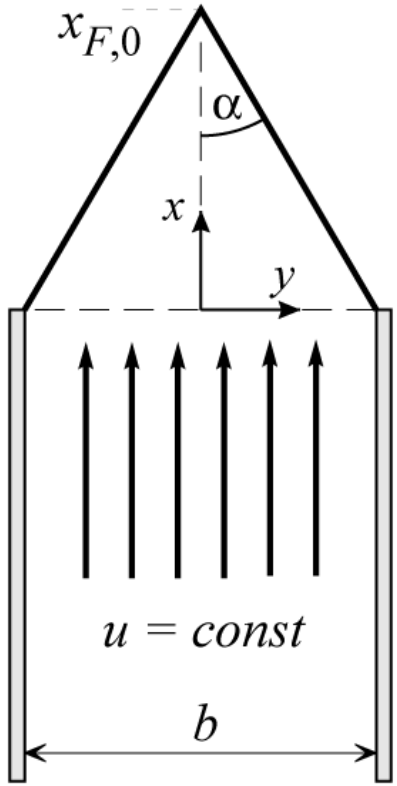
$$u\frac{\partial G}{\partial x} = s_L \sqrt{\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2}$$

With the ansatz $G = x + F(y)$

and $G_0 = 0$ one obtains $u = s_L \sqrt{1 + \left(\frac{\partial F}{\partial y}\right)^2}$

leading to

$$F = \sqrt{\frac{u^2 - s_L^2}{s_L^2}}\,|y| + const.$$



As the flame is attached at $x = 0$, $y = \pm b/2$, where $G = 0$, this leads to the solution

$$G = \sqrt{\frac{u^2 - s_L^2}{s_L^2}}\left(|y| - \frac{b}{2}\right) + x.$$

The flame tip lies with y=0, G = 0 at

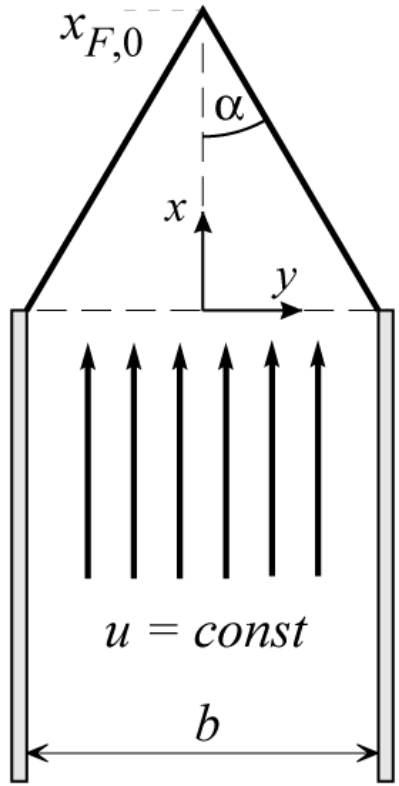
$$x_{F,0} = \frac{b}{2}\sqrt{\frac{u^2 - s_L^2}{s_L^2}}$$

and the flame angle α is given by

$$\tan\alpha = \frac{b}{2x_{F,0}} = \sqrt{\frac{u^2 - s_L^2}{s_L^2}}.$$

With $\tan^2\alpha = \sin^2\alpha/(1-\sin^2\alpha)$ it follows that

$$\sin\alpha = \frac{s_L}{u}$$



which is equivalent to $s_{L,u} = v_{n,u} = v_u \sin\alpha$.

This solution shows a cusp at the flame tip $x = x_{F0}$, $y = 0$. In order to obtain a rounded flame tip, one has to take modifications of the burning velocity due to flame curvature into account. This leads to the concept of flame stretch.

Flame stretch

Flame stretch consists of two contributions: One due to flame curvature and another due to flow divergence.

For a one-step large activation energy reaction and with the assumption of constant properties the burning velocity s_L is modified by these two effects as

$$s_L = s_L^0 - s_L^0 \mathcal{L} \kappa + \mathcal{L} \boldsymbol{n} \cdot \boldsymbol{\nabla} \boldsymbol{v} \cdot \boldsymbol{n}.$$

s^0_L is the burning velocity for an unstretched flame and $\mathcal{L}$ is the Markstein length.

The flame curvature κ is defined as

$$\kappa = \nabla \cdot \boldsymbol{n} = -\nabla \cdot \left(\frac{\nabla G}{|\nabla G|} \right)$$

which may be transformed as

$$\kappa = - \frac{\nabla^2 G + \boldsymbol{n} \cdot \nabla (\boldsymbol{n} \cdot \nabla G)}{|\nabla G|}.$$

The Markstein length $\mathcal{L}$ appearing in

$$s_L = s_L^0 - s_L^0 \mathcal{L} \kappa + \mathcal{L} \boldsymbol{n} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{n}.$$

is of the same order of magnitude and proportional to the laminar flame thickness ℓ_F, their ratio $\ell_F / \mathcal{L}$ is called the Markstein number.

For the case of a one-step reaction with a large activation energy, constant transport properties and a constant heat capacity c_p, the Markstein length with respect to the unburnt mixture reads

$$\frac{\mathcal{L}_u}{\ell_F} = \frac{1}{\gamma}\ln\frac{1}{1-\gamma} + \frac{\mathrm{Ze}(\mathrm{Le}-1)}{2}\frac{(1-\gamma)}{\gamma}\int_0^{\gamma/(1-\gamma)}\frac{\ln(1+x)}{x}dx\ .$$

This expression was derived by [Clavin and Williams (1982)] and [Matalon and Matkowsky (1982)].

Here $\mathrm{Ze} = E(T_b - T_u)/(\mathcal{R}T_b^2)$ is the Zeldovich number, where E is the activation energy and $\mathcal{R}$ the universal gas constant, and Le is the Lewis number of the deficient reactant.

The expression is valid if s_L is defined with respect to the unburnt mixture. A different expression can be derived, if both, s_L and $\mathcal{L}$ are defined with respect to the burnt gas [cf. Clavin, 1985].

We want to explore the influence of curvature on the burning velocity for the case of a spherical propagating flame.

Since the flow velocity is zero in the burnt gas, it is advantageous to formulate the *G*-equation with respect to the burnt gas:

$$\frac{\mathrm{d}r_f}{\mathrm{d}t} = s_{L,b}$$

where $r_f(t)$ is the radial flame position.
The burning velocity is then $s^0_{L,b}$ and the Markstein length is that with respect to the burnt gas $\mathcal{L}_b$.

Here we assume $\mathcal{L}_b > 0$ to avoid complications associated with thermo-diffusive instabilities.

In a spherical coordinate system the *G*-equation reads

$$\frac{\partial G}{\partial t} = s_{L,b}^{0} \left(\left| \frac{\partial G}{\partial r} \right| + \frac{2\mathcal{L}_b}{r} \frac{\partial G}{\partial r} \right),$$

where the entire term in round brackets represents the curvature in spherical coordinates.

We introduce the ansatz

$$G = r_f(t) - r,$$

to obtain at the flame front $r=r_f$

$$\frac{\partial r_f}{\partial t} = s_{L,b}^{0} \left(1 - \frac{2\mathcal{L}_b}{r_f} \right).$$

This equation may also be found in [Clavin (1985)].

This equation reduces to $\frac{\mathrm{d}r_f}{\mathrm{d}t} = s_{L,b}$ for $\mathcal{L}_b = 0$.

It may be integrated to obtain

$$s^0_{L,b} t = r_f - r_{f,0} + 2\mathcal{L}_b \ln\left(\frac{r_f - 2\mathcal{L}_b}{r_{f,0} - 2\mathcal{L}_b}\right),$$

where the initial radius at t=0 is denoted by $r_{f,0}$.

This expression has no meaningful solutions for $r_{f,0} < 2\mathcal{L}_b$ indicating that there needs to be a minimum initial flame kernel for flame propagation to take off.

It should be recalled that

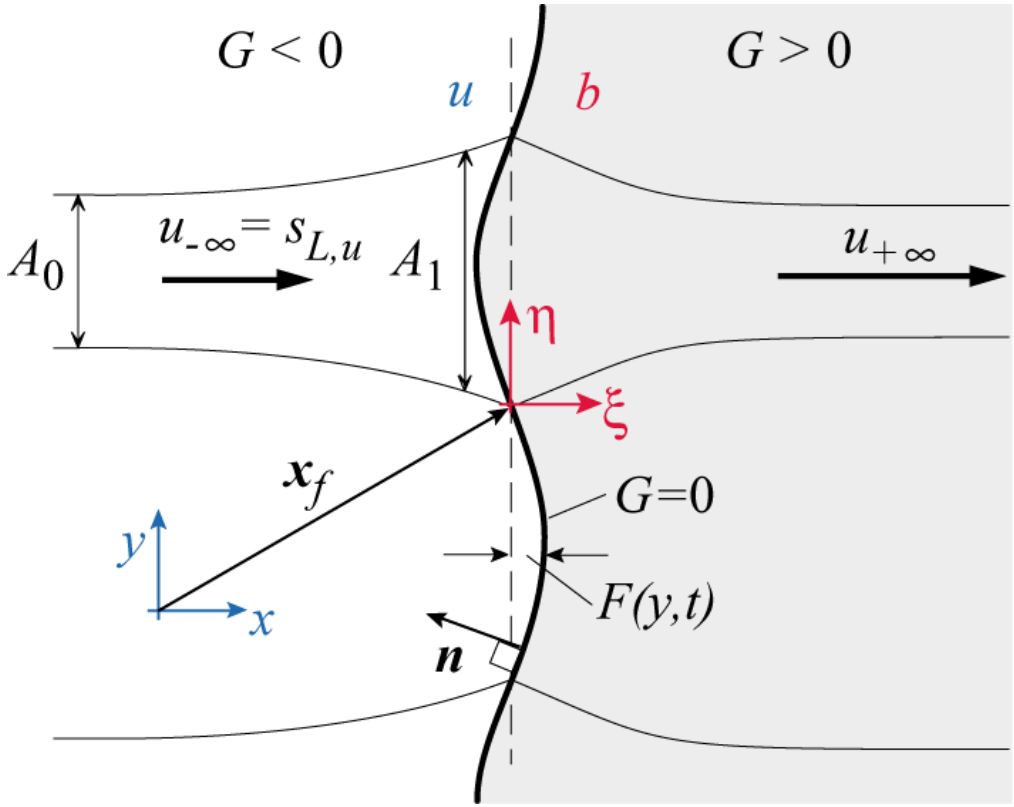
$$s_L = s^0_L - s^0_L \mathcal{L}\kappa + \mathcal{L}\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{v}\cdot\boldsymbol{n}.$$

is only valid if the product $\mathcal{L}\kappa \ll 1$.

For $r_{f,0} > 2\mathcal{L}_b$ curvature corrections are important at early times only.

Flame Front Instability

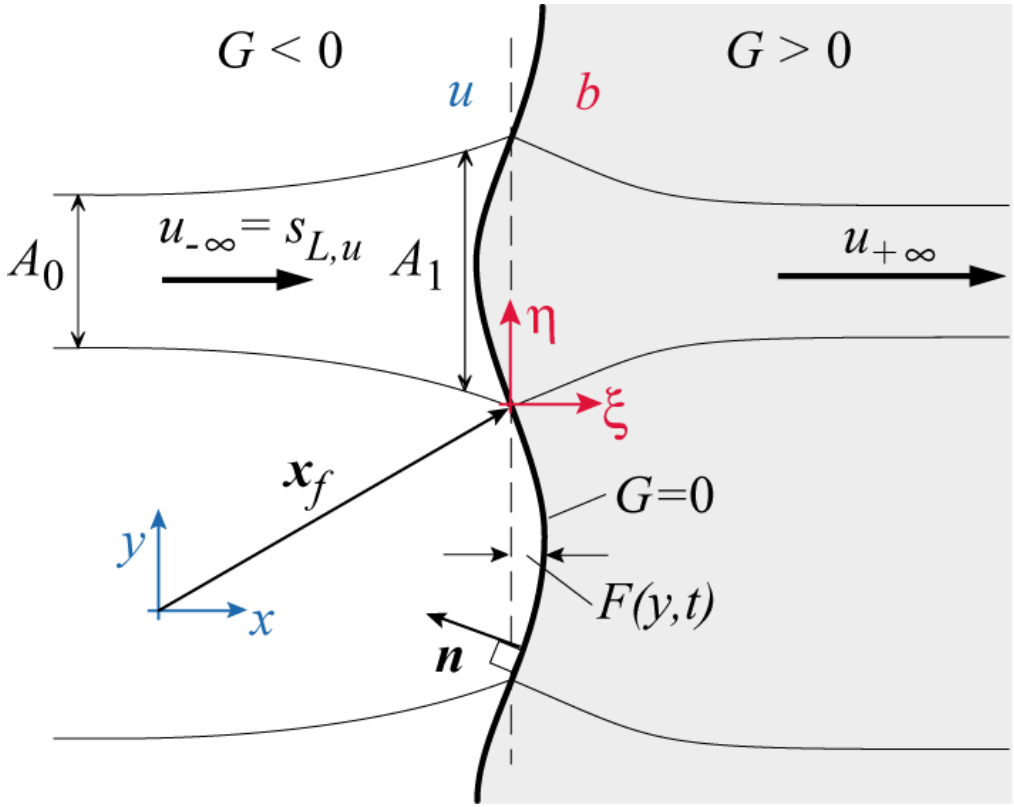
Illustration of the
hydro-dynamic instability
of a slightly undulated flame



Gas expansion in the flame front lead to a deflection of a stream line that enters the front with an angle.

A stream tube with cross-sectional area A_0 and upstream flow velocity $u_{-\infty}$ widens due to flow divergence ahead of the flame.

Expansion at the front induces a flow component normal to the flame contour.
As the stream lines cross the front they are deflected.



At large distances from the front the stream lines are parallel again, but the downstream velocity is

$$u_{+\infty} = (\rho_u/\rho_b)\, u_{-\infty}$$

At a cross section A_1, where the density is still equal to ρ_u the flow velocity due to continuity and the widening of the stream tube is

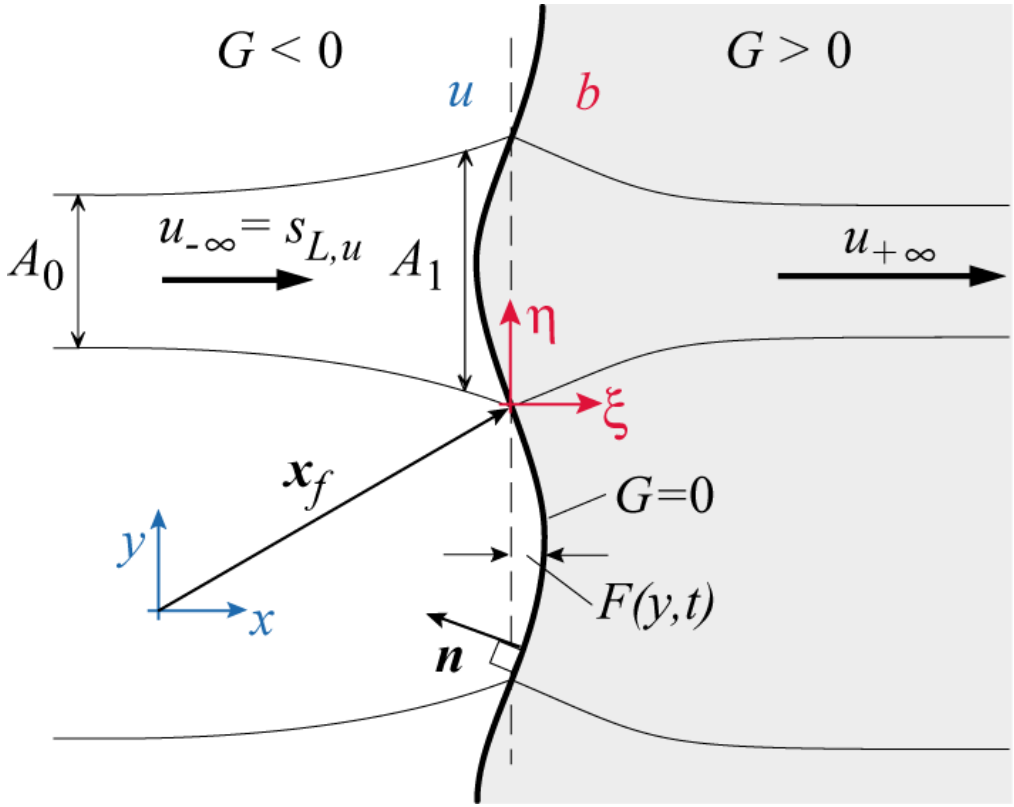
$$u_1 = \frac{A}{A_1}\, u_{-\infty} \leq u_{-\infty}.$$

The unperturbed flame propagates with

$$u_{-\infty} = s_{L,u}$$

normal to itself .

The burning velocity is larger than u_1 and the flame propagates upstream and thereby enhances the initial perturbation.



Simplification:

Viscosity, gravity and compressibility in the burnt and unburnt gas are neglected.

Density is discontinuous at the flame front.

The influence of the flame curvature on the burning velocity is retained,

flame stretch due to flow divergence is neglected.

The burning velocity is given by

$$s_L = s_L^0 \, (1 + \kappa \mathcal{L})$$

Reference values for length, time, density, pressure: $\ell_F \, , \quad \ell_F / s_{L,u} \, , \quad \rho_u \, , \quad \rho_u s_{L,u}^2$

Introduce the density rate: $r = \rho_b / \rho_u < 1$

Dimensionless variables:

$$u^* = u/s_{L,u}, \quad v^* = v/s_{L,u}, \quad p^* = \frac{p}{\rho_u s_{L,u}^2},$$

$$x^* = x/\ell_F, \quad y^* = y/\ell_F, \quad t^* = \frac{t}{\ell_F / s_{L,u}}.$$

The non-dimensional governing equations are then

(with the asterisks removed)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

where $\rho_u = 1$ and $\rho = r$ in the unburnt and burnt mixture respectively.

If G is a measure of the distance to the flame front, the G-field is described by:

$$G = x - F(y,t)$$

With equations

$$\boldsymbol{n} = -\frac{\boldsymbol{\nabla} G}{|\boldsymbol{\nabla} G|}, \qquad \frac{\partial G}{\partial t} - |\boldsymbol{\nabla} G|\, \boldsymbol{n} \cdot \frac{\partial \boldsymbol{x}}{\partial t}\bigg|_{G=G_0} = 0$$

the normal vector $\boldsymbol{n}$ and the normal propagation velocity then are

$$\boldsymbol{n} = \left(-1,\ \frac{\partial F}{\partial y}\right) \Big/ \sqrt{1 + \left(\frac{\partial F}{\partial y}\right)^2}, \quad \boldsymbol{n} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\bigg|_{G=G_0} = \frac{\partial F}{\partial t} \Big/ \sqrt{1 + \left(\frac{\partial F}{\partial y}\right)^2}$$

Due to the discontinuity in density at the flame front, the Euler equations

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y},
\end{aligned}$$

are only valid on either side of the front, but do not hold across it.

Therefore jump conditions for mass and momentum conservation across the discontinuity are introduced [Williams85,p. 16]:

$$(r-1)\,\boldsymbol{n}\cdot\left.\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|_{G=G_0} = \boldsymbol{n}\cdot(r\boldsymbol{v}_+ - \boldsymbol{v}_-)$$

$$(r\boldsymbol{v}_+ - \boldsymbol{v}_-)\,\boldsymbol{n}\cdot\left.\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|_{G=G_0} = \boldsymbol{n}\cdot\left(r\boldsymbol{v}_+\boldsymbol{v}_+ - \boldsymbol{v}_-\boldsymbol{v}_- - (p_+ - p_-)\boldsymbol{I}\right)$$

The subscripts + and - refer to the burnt and the unburnt gas and denote the properties immediately downstream and upstream of the flame front.

In terms of the u and v components the jump conditions read

$$
\begin{aligned}
(r-1)\frac{\partial F}{\partial t} &= ru_+ - u_- - \frac{\partial F}{\partial y}(rv_+ - v_-) \\
(ru_+ - u_-)\frac{\partial F}{\partial t} &= ru_+(u_+ - \frac{\partial F}{\partial y}v_+) - u_-(u_- - \frac{\partial F}{\partial y}v_-) + p_+ - p_- \\
(rv_+ - v_-)\frac{\partial F}{\partial t} &= rv_+(u_+ - \frac{\partial F}{\partial y}v_+) - v_-(u_- - \frac{\partial F}{\partial y}v_-) - \frac{\partial F}{\partial y}(p_+ - p_-).
\end{aligned}
$$

Under the assumption of small perturbations of the front, with $\varepsilon << 1$ the unknowns are expanded as

$$u = U + \epsilon u, \qquad v = \epsilon v$$

$$p = P + \epsilon p, \qquad F = \epsilon f,$$

Jump conditions to leading order

$$U_- = 1, \quad P_- = 0$$

$$U_+ = \frac{1}{r}, \quad P_+ = \frac{r-1}{r},$$

and to first order

$$\begin{aligned}
(r-1)\frac{\partial f}{\partial \tau} &= r u_+ - u_- \\
0 &= 2(u_+ - u_-) + p_+ - p_- \\
0 &= v_+ - v_- + \frac{1-r}{r}\frac{\partial f}{\partial \eta},
\end{aligned}$$

where the leading order mass flux has been set equal to one:

$$\dot{m} = r U_+ = U_- = 1$$

With the coordinate transformation $x = \xi + F(\eta, \tau), \quad y = \eta, \quad t = \tau$

we fix the discontinuity at $\xi = 0$.

To first order the equations for the perturbed quantities on both sides of the flame front now read

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0$$

$$\frac{\partial u}{\partial \tau} + U\frac{\partial u}{\partial \xi} + \frac{1}{\rho}\frac{\partial p}{\partial \xi} = 0$$

$$\frac{\partial v}{\partial \tau} + U\frac{\partial u}{\partial \xi} + \frac{1}{\rho}\frac{\partial p}{\partial \eta} = 0,$$

where $\rho = 1$ for $\xi < 0$ (unburnt gas) and $\rho = r$ for $\xi > 0$ (burnt gas) is to be used.

In case of instability perturbations which are initially periodic in the η-direction and vanish for $\xi \to \pm\infty$ would increase with time.

Since the system is linear, the solution may be written as

$$\boldsymbol{w} = \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \boldsymbol{w}_0 \exp(\alpha\xi) \exp(\sigma\tau - \mathrm{i}k\eta),$$

where σ is the non-dimensional growth rate, *k* the non-dimensional wave number and i the imaginary unit.

Introducing this into the first order equations the linear system may be written as

$$\boldsymbol{A} \cdot \boldsymbol{w} = \boldsymbol{0}$$

The matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \alpha & -\mathrm{i}k & 0 \\ \sigma + \alpha U & 0 & \alpha/\rho \\ 0 & \sigma + \alpha U & -\mathrm{i}k/\rho \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{A}$ are obtained by setting $\det(\boldsymbol{A}) = 0$.

This leads to the characteristic equation

$$\det(\boldsymbol{A}) = \frac{1}{\rho}\left(k^2 - \alpha^2\right)\left(\sigma + \alpha U\right) = 0.$$

Here again $U = 1/r$, $\rho = r$ for $\xi > 0$ and $U = 1$, $\rho = 1$ for $\xi < 0$.

There are three solutions to the characteristic equation for the eigenvalues α_j, $j = 1,2,3$.

Positive values of α_j satisfy the upstream ($\xi < 0$) and negative values the downstream ($\xi > 0$) boundary conditions of the Euler equations.

Therefore

$$\xi > 0: \quad \alpha_1 = -r\sigma, \quad \alpha_2 = -k$$

$$\xi < 0: \quad \alpha_2 = -k.$$

Introducing the eigenvalues into $\boldsymbol{A} \cdot \boldsymbol{w} = 0$ again, the corresponding eigenvectors $\boldsymbol{w}_{0,j}$, j = 1,2,3 are calculated to

$$j = 1: \qquad w_{0,1} = \left(1, \quad \mathrm{i}\frac{\mathrm{r}\sigma}{\mathrm{k}}, \quad 0 \right)$$

$$j = 2: \qquad w_{0,2} = \left(1, \quad \mathrm{i}, \quad -1 + \frac{\mathrm{r}\sigma}{\mathrm{k}} \right)$$

$$j = 3: \qquad w_{0,3} = \left(1, \quad -\mathrm{i}, \quad -1 - \frac{\sigma}{\mathrm{k}} \right)$$

In terms of the original unknowns u, v and the solution is now

$$\xi > 0: \quad \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \left\{ a \begin{pmatrix} 1 \\ \mathrm{i}\frac{r\sigma}{k} \\ 0 \end{pmatrix} \exp(-r\sigma\xi) + b \begin{pmatrix} 1 \\ \mathrm{i} \\ -1 + \frac{r\sigma}{k} \end{pmatrix} \exp(-k\xi) \right\} \exp(\sigma\tau - \mathrm{i}k\eta)$$

$$\xi < 0: \quad \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \quad c \begin{pmatrix} 1 \\ -\mathrm{i} \\ -1 - \frac{\sigma}{k} \end{pmatrix} \exp(k\xi + \sigma\tau - \mathrm{i}k\eta).$$

For the perturbation $f(\eta, \tau)$ the form

$$f = \tilde{f} \exp(\sigma\tau - \mathrm{i}k\eta)$$

will be introduced.

Inserting $$\kappa = -\frac{\boldsymbol{\nabla}^2 G + \boldsymbol{n}\cdot\boldsymbol{\nabla}(\boldsymbol{n}\cdot\boldsymbol{\nabla}G)}{|\boldsymbol{\nabla}G|}, \quad G = x - F(y,t)\,,$$

and

$$u = U + \epsilon u, \quad v = \epsilon v$$

$$p = P + \epsilon p, \quad F = \epsilon f,$$

into the non-dimensional *G*-equation

$$\left(\frac{\partial G}{\partial t} + u\frac{\partial G}{\partial x} + v\frac{\partial G}{\partial y}\right) = \sqrt{\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2}\,(1+\kappa\mathcal{L})$$

satisfies to leading order with

$$u = U + \epsilon u, \quad v = \epsilon v$$

$$p = P + \epsilon p, \quad F = \epsilon f,$$

and $x = 0_-$, $x = 0_+$ respectively.

This leads to first order to

$$u_- = \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial \eta^2}\mathcal{L}$$

$$u_+ = \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial \eta^2}\frac{\mathcal{L}}{r}.$$

With

$$f = \tilde{f}\exp(\sigma\tau - \mathrm{i}k\eta)$$

the jump conditions

$$\begin{aligned}(r-1)\frac{\partial f}{\partial \tau} &= ru_+ - u_- \\ 0 &= 2(u_+ - u_-) + p_+ - p_- \\ 0 &= v_+ - v_- + \frac{1-r}{r}\frac{\partial f}{\partial \eta},\end{aligned}$$

can be written as

$$\begin{aligned}(r-1)\,\sigma\tilde{f} &= r\,(a+b) - c \\ 0 &= 2a + b\,(1 + r\frac{\sigma}{k}) + c\,(\frac{\sigma}{k} - 1) \\ \frac{1-r}{r}\,k\tilde{f} &= a\frac{r\sigma}{k} + b + c\end{aligned}$$

The system

$$u_- = \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial \eta^2}\mathcal{L}$$

$$u_+ = \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial \eta^2}\frac{\mathcal{L}}{r}$$

then reads

$$c = \tilde{f}\,(\sigma + k^2\mathcal{L})$$

$$a + b = \tilde{f}\,(\sigma + \frac{k^2\mathcal{L}}{r})$$

Since equation

$$(r-1)\,\sigma\tilde{f} \;=\; r\,(a+b)-c$$

is linear dependent from equations

$$c = \tilde{f}\,(\sigma + k^2\mathcal{L})$$

$$a+b = \tilde{f}\,(\sigma + \frac{k^2\mathcal{L}}{r})$$

it is dropped and the equations

$$0 \;=\; 2a + b\,(1 + r\frac{\sigma}{k}) + c\,(\frac{\sigma}{k} - 1)$$

$$\frac{1-r}{r}\,k\tilde{f} \;=\; a\frac{\sigma}{k} + b + c$$

and

$$c = \tilde{f}\,(\sigma + k^2\mathcal{L})$$

$$a+b = \tilde{f}\,(\sigma + \frac{k^2\mathcal{L}}{r})$$

remain for the determination of a, b, c and σ(*k*).

Dividing all equations by $k\,\tilde{f}$ one obtains four equations for

$$\hat{a} = a/(k\tilde{f})\,, \quad \hat{b} = b/(k\tilde{f})\,, \quad \hat{c} = c/(k\tilde{f})\,, \quad \varphi = \sigma/k$$

The elimination of the first three unknown yields the equation

$$\varphi^2(1+r) + 2\varphi(1+k\mathcal{L}) + \frac{2k\mathcal{L}}{r} + \frac{r-1}{r} = 0$$

The solution may be written in terms of dimensional quantities as

$$\sigma = \frac{\bar{s_{L0}}k}{1+r}\left\{\sqrt{1+k^2\mathcal{L}^2 - \frac{2k\mathcal{L}}{r} + \frac{1-r^2}{r}} - (1+k\mathcal{L})\right\}$$

Here only the positive root has been taken, since it refers to possible solutions with exponential growing amplitudes.

The relation

$$\sigma = \frac{\bar{s_{L0}} k}{1+r}\left\{\sqrt{1+k^2\mathcal{L}^2 - \frac{2k\mathcal{L}}{r} + \frac{1-r^2}{r}} - (1+k\mathcal{L})\right\}$$

is the dispersion relation which shows that the perturbation *f* grows exponentially in time only for a certain wavenumber range $0 < k < k^*$.

Here k^* is the wave number of which $\varphi = 0$ in

$$\varphi^2(1+r) + 2\varphi(1+k\mathcal{L}) + \frac{2k\mathcal{L}}{r} + \frac{r-1}{r} = 0$$

which leads to

$$k^* = (r-1)/(2\mathcal{L}).$$

For perturbations at wave numbers $k > k^*$ a plane flame of infinitively small thickness, described as a discontinuity in density, velocity and pressure is unconditionally stable.

This is due to the influence of the front curvature on the burning velocity.

As one would expect on the basis of simple thermal theories of flame propagation, the burning velocity increases when the flame front is concave and decreases when it is convex towards the unburnt gas, so that initial perturbations are smoothen.

However, hydrodynamic and curvature effects are not the only influencing factors for flame front stability.

Flame stretch due to flow divergence, gravity (in a downward propagating flame) and the thermo-diffusive effect with a Lewis number larger unity are stabilizing effects.

A more detailed discussion of these phenomena may be found in [Clavin85] and [Williams85].

Exercise

Under the assumption of a constant burning velocity $\underline{s_L = s_{L0}}$ the linear stability analysis leads to the following dispersion relation

$$\sigma = \frac{\bar{s_{L0}} k}{1+r} \left\{ \sqrt{1 + \frac{1-r^2}{r}} - 1 \right\}.$$

Validate this expression by inserting $\mathcal{L} = 0$

$$\sigma = \frac{\bar{s_{L0}} k}{1+r} \left\{ \sqrt{1 + k^2\mathcal{L}^2 - \frac{2k\mathcal{L}}{r} + \frac{1-r^2}{r}} - (1 + k\mathcal{L}) \right\}$$

What is the physical meaning of this result?

What effect has the front curvature on the flame front stability?

Solution

The dispersion relation for constant burning velocity $s_L = s_{L0}$,

$$\sigma = \frac{\bar{s_{L0}} k}{1+r}\left\{\sqrt{1+\frac{1-r^2}{r}}-1\right\}.$$

shows that the perturbation F grows exponentially in time for all wave numbers.

The growth σ is proportional to the wave number k and always positive since the density rate r is less than unity.

This means that a plane flame front with constant burning velocity is unstable to any perturbation.

The front curvature has a stabilizing effect on the flame front stability.

As it is shown in the last section, the linear stability analysis for a burning velocity with the curvature effect retained leads to instability of the front only for the wave number range

$$0 < k < k^* = (r-1)/(2\mathcal{L}),$$

whereas the front is stable to all perturbations with k > k*.