Lecture 10

Flame Instabilities

Hydrodynamic Instability

Darrieus (1938)  Landau (1944)

They treated the flame as a surface of density discontinuity, with a constant flame speed and temperature, similar to the leading order hydrodynamic theory, and examine the stability of a planar flame front.

$$\rho_u / \rho_b = T_b / T_u \equiv \sigma$$

$$\sigma > 1, \text{ thermal expansion}$$
Euler equations
\[ \nabla \cdot \mathbf{v} = 0, \quad \rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g} \]

Flame speed relation
\[ S_f = \mathbf{v} \cdot \mathbf{n} - V_f \bigg|_{\text{lamb.}} = S_L \]

Rankine-Hugoniot Jump conditions
\[
\begin{align*}
[\rho (\mathbf{v} \cdot \mathbf{n} - V_f)] &= 0 \\
[\mathbf{v} \times \mathbf{n}] &= 0 \\
[p + \rho \mathbf{v} \cdot (\mathbf{v} - \mathbf{n} - V_f)] &= 0
\end{align*}
\]
\[
\begin{align*}
[\mathbf{v} \cdot \mathbf{n}] &= (\sigma - 1)S_L \\
[\mathbf{v} \times \mathbf{n}] &= 0 \\
[p] &= -(\sigma - 1)\rho_u S_L^2
\end{align*}
\]

**Plane flame** (in a frame attached to the front)
consider first the case with no gravity

\[
\begin{align*}
\mathbf{v} &= \begin{cases} 
S_L & (x < 0) \\
\sigma S_L & (x > 0)
\end{cases} \\
p - p_0 &= \begin{cases} 
0 & (x < 0) \\
-(\sigma - 1)\rho_u S_L^2 & (x > 0)
\end{cases}
\end{align*}
\]

\[ g > 0 \quad \text{downward propagation} \]
\[ g < 0 \quad \text{upward propagation} \]

We will retain the notation that the top expression in the bracket is for \( x < 0 \) and the bottom expression for \( x > 0 \)
Introduce small disturbances (denoted by prime) and restrict attention to 2D
\[ u = \pi(x) + u'(x,y,t) \quad v = \nu'(x,y,t) \quad p = \pi(x) + p'(x,y,t) \quad f = f(y,t) \]
\[ u'_x + v'_y = 0 \]
\[ \rho u'_i + \rho u'S_i u'_x + \rho u'd'_x = -p'_x \]
\[ \rho v'_i + \rho u'S_i v'_x + \rho u'y'_x = -p'_y \]
\[ u'_x + v'_y = 0 \]
\[ p'_{xx} + p'_{yy} = 0 \]
\[ \rho u'_i + \rho u'S_i u'_x = -p'_x \]

\[ n = \frac{(1,-f_y,-f_z)}{\sqrt{1+f'_y^2+f'_z^2}} \]
\[ V_j = \frac{f_j}{\sqrt{1+f'_y^2+f'_z^2}} \]
\[ n \sim \frac{(1,-f_y)}{f_i} \quad t \sim \frac{f_i}{1} \]
\[ V_j \sim f_i \]
\[ x = f(y,z,t) \]
\[ f_i = u'(0^-) \]
\[ \rho_b \]
\[ \rho_u \]
\[ \sqrt{1+f'_y^2+f'_z^2} \]
\[ \sqrt{1+f'_y^2+f'_z^2} \]
\[ \sqrt{1+f'_y^2+f'_z^2} \]
Solve in the regions $x < 0$ with $\rho = \rho_u$ and $x > 0$ with $\rho = \rho_b$

\[
\begin{align*}
    u' + v' &= 0 \\
    p'_{xx} + p'_{yy} &= 0 \\
    \rho u' + \rho_b S_L u' &= -p'
\end{align*}
\]

Jump conditions across $x = 0$

\[
\begin{align*}
    [u'] &= 0 \\
    [v'] &= -(\sigma - 1) S_L f_y \\
    [p'] &= 0 \\
    f_x &= u'(0^-)
\end{align*}
\]

These conditions are to be satisfied across the flame front, i.e. across $x = f(x,t)$, but since $f << 1$ they can be related to values across $x = 0$ using a Taylor expansion. Since the basic state solution is a piecewise constant function, the expansion of $G(x=f) \sim G(0)$.

---

Normal mode analysis

\[
\begin{align*}
    v' &= V(x) e^{iky + \omega t} \\
    p' &= P(x) e^{iky + \omega t} \\
    f' &= A e^{iky + \omega t}
\end{align*}
\]

This is the method of normal modes, where it is assumed that an arbitrary disturbance introduced as an I.C., can be expanded in a Fourier integral (or series) over $k$. Being a linear problem it suffices to examine the response to one Fourier component.

In three-dimensions, disturbances are of the form $f' \sim e^{i k \cdot z + \omega t}$, where $k = (k_1, k_2)$ is the wave-vector and $z = (y, z)$. The 2D results (in this case) can be easily extended to 3D if

\[ k = |k| = \sqrt{k_1^2 + k_2^2} \]
We are interested in the behavior of the solution as $t \to \infty$.

$$\omega = \omega_R + i \omega_I$$

$$f = A e^{iky + \omega t} = A e^{(ky + \omega_I)t} e^{\omega_R t}$$

If the perturbation goes to zero as $t \to \infty$, which is assured when $\omega_R < 0$ for all values of $k$, the basic state (i.e., the planar flame) is stable.

For instability it is sufficient that $\omega_R > 0$ for one mode (one value of $k$)

When substituting into the system of PDEs, the problem reduces to a set of ODEs for the determination of $V(x), P(x), A$, with the real valued $k$, and the (unknown) complex-valued $\omega$ as parameters.

\[
\begin{align*}
P_{xx} - k^2 P &= 0 \\
\rho \omega U + \rho_s S_L U_x &= -P_x \\
U_x + ikV &= 0
\end{align*}
\]

\[
\begin{align*}
[U] &= 0 \\
[V] &= -ik(\sigma - 1)S_L A \\
[P] &= 0
\end{align*} \quad \text{at } x = 0
\]

\[
\begin{align*}
[U] &= 0 \\
[V] &= -i\omega(\sigma - 1)S_L A \\
[P] &= 0
\end{align*} \quad \text{at } x = 0
\]

\[
U(0^-) = \omega A
\]
The associated boundary conditions are homogeneous. It is, therefore a homogeneous problem with homogeneous BCs, for which the trivial solution $U = V = P = A = 0$ is a solution. The question is whether there are other nontrivial solutions; it is, therefore, an eigenvalue problem with $\omega$ the eigenvalue.

In the present problem the only parameters are $\sigma, S_L$, in addition to $k$ and $\omega$. Solution to the eigenvalue problem exists only if

$$F(\sigma, S_L; k, \omega) = 0$$

$\omega = \omega(\sigma, S_L; k)$ dispersion relation

Dimensional considerations show that

$$\omega = \omega_{DL}(\sigma) S_L k$$

The dimensionless factor $\omega_{DL}(\sigma)$ and, in particular its sign, remain to be determined.

It is instructive to examine first, the case for which $\sigma - 1 \ll 1$

$$\sigma - 1 \equiv \varepsilon \ll 1$$

$$V = \varepsilon \tilde{V}, \quad U = \varepsilon \tilde{U}, \quad P = \varepsilon \tilde{P}, \quad \omega = \varepsilon \tilde{\omega}$$

$$\begin{align*}
P_{xx} - k^2 P &= 0 \\
\rho_0 U + \rho_0 S_L U_x &= -P_x \\
U_x + ikV &= 0
\end{align*}$$

$$\begin{align*}
\tilde{P}_{xx} - k^2 \tilde{P} &= 0 \\
\rho_0 S_L \tilde{U}_x &= -\tilde{P}_x \\
\tilde{U}_x + ik\tilde{V} &= 0
\end{align*}$$

$$\begin{align*}
[U] = 0 \\
[V] = -ik(\sigma - 1)S_L A \\
[P] = 0
\end{align*}$$

at $x = 0$

$$\begin{align*}
[\tilde{U}] = 0 \\
[\tilde{V}] = -ikS_L A \\
[\tilde{P}] = 0
\end{align*}$$

at $x = 0$

$$U(0^-) = \omega A$$

$$\tilde{U}(0^-) = \tilde{\omega} A$$
There is a discontinuity in the transverse velocity along \( x = 0 \). Hence, the disturbed flame is equivalent to a flat vortex sheet (at the mean position \( x = 0 \)) of strength proportional to \( -f_y \).

A surface across which the tangential velocity changes abruptly is a vortex sheet; a sheet where an infinite amount of vorticity is concentrated (e.g., a boundary layer).

Note that there is no vorticity elsewhere; \( V_x - U_y = 0 \) everywhere at this order. When \( \sigma - 1 = O(1) \) there is vorticity in the burned gas, and the vorticity generated at the flame is transported downstream.
the flame front is convected by the flow

the disturbed flame is equivalent to a flat vortex sheet at \( x = 0 \) of strength \(- (\sigma - 1) f_t\) that always increases in time (because \( \omega_R > 0 \))

The concentrated vorticity, through the Biot-Savart law induces an axial velocity \( u \) proportional to \( f_t \) that convects segments of the flame intruding towards the burned gas further upstream and those intruding towards the unburned gas further downstream.

In general, for arbitrary \( \sigma \)

\[
\begin{align*}
P_x - k^2 P &= 0 \\
\rho_0 U + \rho S_t U_x &= -P_x \\
U_x + i k V &= 0
\end{align*}
\]

Jump conditions

\[
\begin{pmatrix}
1 + \dot{\omega} & (\dot{\omega}/\sigma) - 1 & 0 \\
1 - (\sigma - 1)/\dot{\omega} & 1 & \dot{\omega}/\sigma \\
1 & -1 & -1
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2 \\
C_3
\end{pmatrix} = 0
\]

\[
(\sigma + 1)\dot{\omega}^2 + 2\dot{\omega} - \sigma(\sigma - 1) = 0
\]
\[(\sigma + 1)\dot{\omega}^2 + 2\sigma\dot{\omega} - \sigma(\sigma - 1) = 0\]

\[\omega = \frac{-2\sigma \pm \sqrt{4\sigma^2 + 4\sigma(\sigma^2 - 1)}}{2(\sigma + 1)}\]

For \(\sigma > 1\), one of the two roots is always positive.

\[\omega = \frac{1}{\sigma + 1} \left\{\sqrt{\sigma^3 + \sigma^2 - \sigma}\right\} S_L k\]

Plane flames are unconditionally unstable.

This is the celebrated Darrieus-Landau (DL) or hydrodynamic instability.

- Short waves \((k >> 1)\) grow faster than long waves \((k << 1)\)
- But the result is not valid for short enough waves \(\lambda = 2\pi/k \sim D_\text{th}/S_L\)
- Thermal expansion is responsible for the instability
- For weak thermal expansion \(\omega \sim (\sigma - 1)S_L k/2\)

The hydrodynamic instability appears to contradict the fact that planar flames have been observed in the laboratory. Although there may be stabilizing influences of diffusion at the shortwave range, as we shall see, the instability, which is a result of thermal expansion is always present, and is the dominant phenomenon in large scale flames, where diffusion play a limited role.

**The DL instability has many ramifications in combustion.**
Formation of cusps and crests

Since the flame propagates normal to itself at a constant speed, successive locations of the front can be constructed geometrically using Huygen’s principle. The displacement over a time interval \( t \) is constructed from a family of spheres of radius \( S_L t \) centered on the flame surface. The envelope draws the new location of the flame front.

The convex/concave sections of the flame behave differently during the propagation. The convex sections (towards the unburned gas) of the flame expand so that a possible break in the front heals itself. The concave sections contract forming cusps.

![Diagram showing the formation of cusps and crests](image)

V-shaped flame in a weakly turbulent stream
Bunsen burner flames

Bunsen flame experiments show that as the flow rate through the burner is reduced the opening angle of the wedge flame becomes smaller. One might expect that when the incoming velocity $U$ approaches $S_L$ the flame will eventually become flat. Instead the flame adopts a different shape with two sharp crests instead of one. The hydrodynamic instability does not permit flames that are too flat.

![Flame Shapes](image)

**Fig. 5.** Change of flame shape with decreasing volume flow (from left to right).

Uberoi et al., POF 1958
Flames in tubes

Flames in tubes are generally curved (convex towards the unburned gas)

Gravity effects

\[ v = i \left[ \frac{S_L}{\sigma \Omega_L} + V(x) e^{iky + \omega t} \right] \]
\[ p - p_0 = \left\{ \begin{array}{ll}
- \rho_b g \varepsilon_x + P(x) e^{iky + \omega t} \\
- (\sigma - 1) \rho_b S_L^2 - \rho_b g \varepsilon_x + P(x) e^{iky + \omega t}
\end{array} \right. \]

The only difference from the previous discussion is in the jump condition for the pressure.

\[ [p]_{x-f} = -(\sigma - 1) \rho_b S_L^2 \]
\[ [p]_{x-f} = [p]_{x-0} + \left( \frac{\partial p}{\partial x} \right)_{x-0} f + \ldots \]
\[ -(\sigma - 1) \rho_b S_L^2 + [P]_{x-0} e^{ikx} + (\rho_v - \rho_b) g f = -(\sigma - 1) \rho_b S_L^2 \]
\[ [P]_{x-0} = -\frac{\sigma - 1}{\sigma} \rho_v g A \]
\[
\begin{pmatrix}
1 + \frac{\sigma^{-1} Fr^{-1}}{\sigma} (\hat{\omega}/\sigma) - 1 & 0 & C_1 \\
1 - (\sigma - 1)/\hat{\omega} & 1 & \hat{\omega}/\sigma & C_2 \\
1 & -1 & -1 & C_3
\end{pmatrix}
= 0
\]

Fr = \frac{k S_L^2}{g}

Froude number

\[(\sigma + 1)\hat{\omega}^2 + 2\sigma\hat{\omega} - (\sigma - 1)(\sigma - Fr^{-1}) = 0\]

\[
\omega = \frac{1}{\sigma + 1} \left\{ \sqrt{\sigma^3 + \sigma^2 - \sigma} \right\} S_L k + \ldots \quad \text{for } k \gg 1
\]

Influence of gravity dies out as \( k \to \infty \); short waves remain unstable

\[
\omega \sim \frac{1}{\sigma + 1} \left\{ \sqrt{\sigma^3 + \sigma^2 - \sigma} \right\} S_L k + \ldots \quad \text{for } k \ll 1
\]

Long waves are stabilized by gravity for downward propagation (\( g > 0 \)) but remain unstable for upward propagation (\( g < 0 \)).

**downward propagation** (\( g > 0 \))

The flame is stable to disturbances with \( k < g/\sigma S_L^2 \)

- gravity stabilizes the long waves (\( \lambda > 2\pi \sigma S_L^2/g \))
- the range of stable wavelengths \( \sim S_L^2 \)

**upward propagation** (\( g < 0 \))

The flame is unconditionally unstable; gravity acts to further destabilize the flame
Diffusive -Thermal Instabilities

Pure diffusive-thermal instabilities can be studied by filtering out the hydrodynamic disturbances. This is most easily done by setting $\rho = \text{const.}$

The flow field is unperturbed and remains (in a frame attached to the flame) constant and uniform with $u = S_L$.

\[
\rho c_p \frac{DT}{Dt} - \lambda \nabla^2 T = QBpY \exp(-E/RT)
\]
\[
\rho \frac{DY}{Dt} - (\rho D) \nabla^2 Y = -BpY \exp(-E/RT)
\]

reactive-diffusive system

The only parameter is the reduced Lewis number $\ell = (Le - 1)\beta$ where $\beta$ is the Zel’dovich number

\[
b_0\omega^3 + b_1\omega^2 + b_2\omega + b_3 = 0
\]
\[
b_i = b_i(\ell, k)
\]

Sivashinsky, CST 1987

The dispersion relation

\[
f = Ae^{iky+\omega_1t} e^{\omega_R t} = Ae^{iky+\omega_R t}
\]

marginal states correspond to $\omega_R = 0$

$\omega_1 = 0 \Rightarrow f = Ae^{iky}$ cellular flames

$\omega_1 \neq 0 \Rightarrow \begin{cases} k = 0 \Rightarrow f = Ae^{i\omega_1 t} & \text{planar pulsating flames} \\ k \neq 0 \Rightarrow f = Ae^{i(k(y+ct))} & c = \omega_1/k \end{cases}$

traveling waves along the flame surface

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Conditions for stability:

\[ 2 + 8k^2 + \ell > 0 \]

\[ 128(2k)^4 + (-3\ell^2 + 16\ell + 128)(2k)^2 + (32 + 8\ell - \ell^2) > 0 \]

Stable

Unstable

Steady cellular flames

Pulsating and/or travelling waves

Cellular flames

Thermal diffusivity < Mass diffusivity

\[ Le < 1 \]
The critical Lewis number for cellular flames: 

\[ (Le - 1) \frac{E(T_v - T_a)}{RT_v} = -2 \]

**Hydrocarbon-air flames**

- \[ D_{th}(N_2) = 0.19 \text{ cm}^2/\text{s} \]
- \[ D_{CH_4-N_2} = 0.11 \text{ cm}^2/\text{s} \]
- \[ D_{O_2-N_2} = 0.22 \text{ cm}^2/\text{s} \]

**Hydrogen-air**

- \[ D_{th}(N_2) = 0.19 \text{ cm}^2/\text{s} \]
- \[ D_{H_2-N_2} = 0.61 \text{ cm}^2/\text{s} \]

The range of Lewis numbers, from rich to lean mixtures

- \[ Le \sim \{ 0.86, \ 1.73 \} \]
- \[ \ell \sim \{-3, \ 11\} \]

Lean H_2–air mixture

- \[ Le \sim 0.31 \]
- \[ \ell \sim -10 \]

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Quinard, 1990

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The critical Lewis number for pulsating flames: 

\[
(Le - 1) \frac{E(T_b - T_u)}{RT_b^2} = 10
\]

The critical Lewis number for the onset of oscillations is much larger than one, implying that oscillations are likely to be observed in lean mixtures of heavy fuels, or rich mixtures of light fuels. The critical value, however, is quite large and is not likely to be reached in common combustion mixtures.

In the presence of heat loss the critical Lewis number is reduced significantly and the instability may be accessible. Indeed it has been observed in porous plug burners when the flame was sufficiently close to the burner.

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**Combined effects of hydrodynamic and diffusive-thermal Instabilities**

The early work of Markstein in the 1950's was phenomenological in nature, extending the Darrieus-Landau results by assuming a dependence of the flame speed on curvature.

The more rigorous results are based on the hydrodynamic asymptotic theory, which includes the influences of diffusion resulting in the internal flame structure assumed thin but of finite thickness, and extends the DL growth rate

\[
\omega = \frac{\sqrt{\sigma^3 + \sigma^2 - \sigma - \sigma}}{\sigma + 1} S_L k
\]

\[
\omega_{DL}
\]

to include higher order terms in \(k\).
Planar Flames
Pelce & Clavin, (JFM, 1982); Matalon & Matkowsky (JFM, 1982); Frankel & Sivashinsky (CST, 1982)

\[ \omega = \omega_{DL} S_1 k - \lambda_1 \left[ B_1 + \beta (Le_{eff} - 1) B_2 + Pr B_3 \right] S_1 k^2 \]

- DL instability
- Heat conduction stabilizing
- Viscous effects stabilizing
- Species diffusion stabilizing

The coefficients \( \omega_{DL}, B_1, B_2, B_3 > 0 \) depend only on thermal expansion \( \sigma \).
Recall \( Le_{eff} \) is a weighted average of the fuel and oxidizer Lewis numbers with a heavier weight on the deficient component.

The short wavelength disturbances \( (\lambda > \lambda_c = 2\pi/k_c) \) are stabilized by diffusion

\[ Le_{eff} < Le_{eff}^* \]
- The short wavelength disturbances are also unstable
- Hydrodynamic instability is enhanced by diffusion effects

\[ Le_{eff} > Le_{eff}^* \]
- The short wavelength disturbances \( (\lambda > \lambda_c = 2\pi/k_c) \) are stabilized by diffusion

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Hydrodynamically unstable flame

$\mathcal{L} = 0.075$ corresponding to $Le > Le^*$, for $\sigma = 4$
critical wavelength $\lambda_c = 1.7$

steady propagation: $x = -Ut + f(y)$

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**Spherical Flames** \((R \gg l_f)\)

Rechtien & Matalon, (C&F, 1987); Addabbo et al. (ProCI, 2002)

\[ r = R(t) [1 + A(t) S_n(\theta, \phi)] \]

**growth-rate:**
\[ \frac{1}{A} \frac{dA}{dt} = \frac{\dot{R}}{R} \left( \omega - \frac{l_f}{R} \right) \]

\[ \frac{1}{A} \frac{dA}{dt} = \frac{\dot{R}}{R} \left\{ \frac{\Omega}{l_f} - \frac{l_f}{R} \left[ \dot{B}_1 + \beta (L_{\text{eff}} - 1) \dot{B}_2 + \text{Pr} \dot{B}_3 \right] \right\} \]

- **hydrodynamic**
- **diffusion**

---

\( \text{Le}_{\text{eff}} > \text{Le}^*_{\text{eff}} \) The amplitude \( A \) first decay, but after the flame reaches a certain critical size \( R_c \), it starts growing in time. The instability is hydrodynamic in nature.

\( \text{Le}_{\text{eff}} < \text{Le}^*_{\text{eff}} \) The amplitude grows in time immediately upon its incept. The flame is diffusively unstable.
Growth rate computed for a lean propane/air flame ($\sigma = 5.9, \beta(Le_{eff} - 1) = 4.93$)

Marginal Stability Curves

Lean propane/air: $\sigma = 5.9, \ Le_{eff} = 1.49$

Growing modes:

$\Lambda_{min} < \Lambda < \Lambda_{max}$

$\Lambda_{max} = 2\pi R/n \sim R$

$\Lambda_{min} = 2\pi R/n \sim \text{const.}$
Bradley et al. (C&F, 2000) Iso-octane ($\phi = 1.4 - 1.6$) Methane ($\phi = 1.0$)

Fig. 11. Experimental wave numbers, $N$, as a function of $Pe$. + cine films of cells (between fractured cracks), $\times$ PLIF images (between fractured cracks), $\bigcirc$ methane-air PLIF images (no fracturing), $\bullet$ PLIF images (no fracturing).

Bradley et al. (C&F, 2000)

- Iso-octane ($\phi = 1.4 - 1.6$)
- Methane ($\phi = 1.0$)

cellular (diffusive-thermal) lean hydrogen-air flame

$Le_{\text{eff}} < 1$

inherently stable lean butane-air flame

$Le_{\text{eff}} > 1$

Strehlow, 1969
Hydrogen-air spherical flames

Law, 2000

\( \phi = 0.6, \ 5 \text{ atm} \)

\( \phi = 4, \ 5 \text{ atm} \)

\( \phi = 4, \ 20 \text{ atm} \)

3.5 ms

2.5 ms

3 ms

\( Le < Le^* \)

Diffusive-thermal instability

\( Le > Le^* \)

Hydrodynamic instability

R, lowered as a result of increasing the pressure (a thinner flame)