Lecture 12
Droplet Combustion
Spray Modeling

Spray combustion:

Many practical applications liquid fuel is injected into the combustion chamber resulting in fuel spray.

Spray combustion involves many physical processes, including atomization, droplet collision and agglomeration, vaporization, heat and mass transfer, droplet-air and vapor-air mixing, ignition, turbulence, premixed and/or diffusion flames, pollutant production and flame extinction.

The liquid fuel is atomized by the combined action of aerodynamical shear, strain and surface tension producing a large number of small droplets which increase the overall surface area exposed to the oxidizer and enhance the rates of heat and mass transfer.

The atomization of a 1 cm diameter droplet of liquid into droplets of 100 µm diameter, for example, produces a million droplets and increases the overall surface area by a factor of $10^4$. 
The droplets distribution range from few microns to around 500\(\mu\)m, primarily because practical nozzles cannot produce sprays of uniform drop size at typical operating conditions. Furthermore, many of the larger droplets produced in the initial liquid disintegration undergo further breakup into smaller droplets.

Depending on the droplet distribution, the mode of operation and the state of the ambience, different regimes of burning occur, including:

- a diffusion flame lies outside the cloud of droplets burning the ambient oxygen with the fuel that originates from the vaporizing droplets
- droplet burning occurring all throughout the cloud or in a layer on the edge of the cloud,
- internal group combustion with an internal flame that separates a group of vaporizing droplets from a group of individually burning droplets
- individual droplet combustion where individual droplets burn with oxygen diffusing across the resulting cloud of burning droplets
**Droplet Vaporization and Combustion:**

**Assumptions:**
- droplet is a sphere
- spherical symmetry
- single component fuel
- uniform properties inside the droplet
- no fluid motion inside the droplet
- quiescent ambience
- no gravity
- quasi-steady approximation
- one-step overall chemical reaction (F + O → Products)

\[
\dot{m} = 4\pi r_s^2 \rho_s \frac{dr_s}{dt}
\]

Overall mass conservation of the droplet

\[
-\frac{d}{dt} \left( \frac{4}{3}\pi \rho_s r_s^3 \right) = \dot{m}
\]

mass flow rate

\[
\frac{dr_s}{dt} = \frac{4\pi r_s^2 \rho_s (v - \dot{r}_s)}{\dot{m}}
\]

mass per unit time leaving the droplet

Since \( \rho_g/\rho_l \ll 1 \) (typically \( \sim 10^{-2} - 10^{-3} \)) the droplet recedes very slowly compared to the gas-phase processes, and

\[
\frac{dr_s}{dt} \sim -\frac{\rho_g}{\rho_l} v
\]

a direct consequence of the jump relation \( \left[ \rho (v \cdot n - V_I) \right] = 0 \) across the interface \( r = r_s \), namely \( -\rho_l \dot{r}_s = \rho_g (v - \dot{r}_s) \) after using \( \rho_g/\rho_l \ll 1 \).

**quasi-steady approximation**

\( \dot{m} \) or \( \rho_g v \) can be determined from the steady equations in the gas phase, with the droplet history determined a-posteriori from the relation above.
Conditions for the mass fractions at the liquid-gas interface can be derived by integrating the species equations across the moving liquid-gas interface, \( r = r_s \), or using the previously derived jump condition

\[
\left[ \rho Y_i ((v + V_i) \cdot n - V_i) \right] = 0
\]

where we have assumed that no reaction occurs along the interface

\[
\rho_g Y_i (v - \dot{r}_s) - \rho_g D_i \frac{dY_i}{dr} = -\rho_l Y_i \dot{r}_s
\]

Using the approximation \( \dot{r}_s \sim (\rho_g/\rho_l)v \)

\[
\begin{align*}
\rho_g v Y_F - \rho_g D_F \frac{dY_F}{dr} &= \rho_g v \\
\rho_g v Y_O - \rho_g D_O \frac{dY_O}{dr} &= 0
\end{align*}
\]

Similarly, integrating the energy (expressed in terms of enthalpy) across the moving liquid-gas interface, \( r = r_s \), or using the previously derived jump condition

\[
\left[ \rho h (v \cdot n - V_I) + p V_I + q \cdot n \right] = 0
\]

\[
\rho_g h_g (v - \dot{r}_s) - \lambda_g \frac{dT}{dr}_{r_s} = -\rho_l h_l \dot{r}_s - \lambda_l \frac{dT}{dr}_{r_s}
\]

where we have assumed that no reaction occurs along the interface and neglected kinetic energy and viscous dissipation in the energy equation (the small Mach number approximation). Then, using the approximation \( \dot{r}_s \sim (\rho_g/\rho_l)v \)

\[
\rho_g v (h_g - h_l) - \lambda_g \left( \frac{dT}{dr}_{r_s} - \frac{dT}{dr}_{r_s} \right) = 0
\]

latent heat of vaporization
\[ \rho \nu \lambda - \lambda \left( \frac{dT}{dr} \right)_{a^+} - \frac{dT}{dr} \right)_{a^-} = 0 \]

assuming all the heat conducted inwards goes to heating the liquid fuel,

\[ 4\pi a^2 \lambda \left( \frac{dT}{dr} \right)_{a^-} \approx 4\pi a^3 \rho c_p \frac{dT}{dt} \]

\[ \lambda \frac{dT}{dr} = \rho \nu Q_\nu + \frac{a}{3} \rho c_p \frac{dT}{dt} \]

often neglected.

Alternatively it may be accounted for by replacing \( Q_\nu \) with an “effective” value that accounts for internal heating.

\[ \lambda \frac{dT}{dr} = \rho \nu Q_\nu \]

In the following, the gas density will be denoted by \( \rho \) (with no subscript) and the droplet radius \( a \).

The steady conservation equations are

\[ \frac{d}{dr} (r^2 \rho v) = 0 \]

\[ \rho_0 \frac{dY_F}{dr} - \rho \frac{d\rho}{dr} \left( \frac{1}{r^2} \frac{d}{dr} \right) \left( r^2 \frac{dY_F}{dr} \right) = -\nu_F W_F \omega \]

\[ \rho_0 \frac{dY_O}{dr} - \rho \frac{d\rho}{dr} \left( \frac{1}{r^2} \frac{d}{dr} \right) \left( r^2 \frac{dY_O}{dr} \right) = -\nu_O W_O \omega \]

\[ \rho \frac{d^2 T}{dr^2} + \frac{1}{\nu} \frac{d}{dr} \left( \frac{1}{r^2} \frac{d}{dr} \right) = 0 \]

\[ \rho \frac{d}{dr} \left( \frac{1}{r^2} \frac{d}{dr} \right) \left( r^2 \frac{d^2 T}{dr^2} \right) = \frac{Q}{c_p} \]

\[ \rho T = \frac{P_0}{\mathcal{W}} \]

\[ \omega = B \left( \frac{\rho Y_F}{W_F} \right) \left( \frac{P_0}{W_O} \right) e^{-E/R_T} \]

Conditions far away, as \( r \to \infty \)

\[ T = T_\infty, \quad \rho = \rho_\infty, \quad Y_F = 0, \quad Y_O = Y_{O_\infty} \]
If one assumes that the phase change at the interface occurs very fast, equilibrium conditions prevail and a definite relation exists between the vapor partial pressure and the droplet surface temperature. This relation is of the form

\[ pY_F = k \exp \left( \frac{Q_v}{R} \left( \frac{1}{T_B} - \frac{1}{T} \right) \right) \]

known as the **Claussius Clapeyron relation**. Instead we shall use for simplicity that the surface temperature is constant, and at \( r = a \)

\[ T = T_B \]

Boundary conditions at \( r = a \)

\[ \rho v Y_F - \rho D_F \frac{dY_F}{dr} = \rho v \]
\[ \rho v Y_O - \rho D_O \frac{dY_O}{dr} = 0 \]
\[ \lambda \frac{dT}{dr} = \rho v Q_v \]
\[ T = T_B \]

We write dimensionless equations using \( a \) as a unit of distance, \( D_{th}/a \) as a unit of velocity and normalize the density and temperature with respect to their ambient values \( \rho_\infty \) and \( T_\infty \).

We also use \( M = \frac{\dot{m}/4\pi}{a(\lambda/c_p)} \) to denote the dimensionless mass flux.

We have a 6th-order system, with 7 BCs; the extra one for the determination of \( M \).

at \( r = 1 \):

\[ MY_F - Le_F^{-1} \frac{dY_F}{dr} = M \]
\[ MY_O - Le_O^{-1} \frac{dY_O}{dr} = 0 \]
\[ \frac{dT}{dr} = ML_v, \quad T = T_s \]

as \( r \to \infty \)

\[ T = \rho = 1, \quad Y_O = Y_{O_\infty} \quad Y_F = 0 \]
The parameters are
\[ \nu = \nu_O W_O / \nu_F W_F \quad \beta_0 = E / RT_\infty \]
\[ q = Q / \nu_F W_F c_p T_\infty \quad D = \frac{\mu^2 \alpha a^2 \nu_F B}{(\lambda/c_p) W_O} \]
\[ L_v = Q_v / c_p T_\infty \quad T_s = T_B / T_\infty \]

We note in particular, the dependence on the Damköhler number on the droplet radius \( D \sim a^2 \) and on the ambient pressure \( D \sim P_0 \).

We also assume unity Lewis numbers.

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Pure vaporization: \( D = 0 \)

\[ Y_F = 1 - e^{-M/r} \]
\[ Y_O = Y_{O \infty} e^{-M/r} \]
\[ T = T_s - L_v + L_v e^{M(1-1/r)} \]

In dimensional form, the vaporization rate is
\[ \dot{m} = \frac{4\pi a \lambda}{c_p} \ln \left( 1 + \frac{c_p(T_\infty - T_B)}{Q_v} \right) \]

transfer number \[ B_v = \frac{c_p(T_\infty - T_B)}{Q_v} = \text{impetus for transfer} / \text{resistance to transfer} \]

Clearly \( B_v > 0 \), which implies that, for vaporization, \( T_\infty > T_B \).
Combustion:
We will consider the Burke-Schumann limit $D \to \infty$ and make use of the coupling functions.

\[ T + qY_F = 1 + (T_s - 1 + q - L_v)(1 - e^{-M/r}) \]
\[ T + q\nu^{-1}Y_O = (T_s - L_v)(1 - e^{-M/r}) + (1 + q\nu^{-1}Y_{O\infty})e^{-M/r} \]

from which we can also write

\[ Y_F - \nu^{-1}Y_O = 1 - (1 + \nu^{-1}Y_{O\infty})e^{-M/r} \]

The profiles are readily available

\[
Y_F = \begin{cases} 
1 - (1 + \nu^{-1}Y_{O\infty})e^{-M/r} & (r < r_f) \\
0 & (r > r_f) 
\end{cases}
\]

\[
Y_O = \begin{cases} 
0 & (r < r_f) \\
(\nu + Y_{O\infty})e^{-M/r} - \nu & (r > r_f) 
\end{cases}
\]
\[ T = \begin{cases} (T_s - L_v)(1 - e^{-M/r}) + (1 + q\nu^{-1}Y_{O\infty})e^{-M/r} & (r < r_f) \\ 1 + (T_s - 1 + q - L_v)(1 - e^{-M/r}) & (r > r_f) \end{cases} \]

There are few conditions that remain to be satisfied; these are \( T = T_s \) at \( r = 1 \), and continuity of all variables at \( r = r_f \). The first provides and expression for \( M \) and the others determine the flame standoff distance and the flame temperature.

\[ M = \ln \left(1 + \frac{1 + q\nu^{-1}Y_{O\infty} - T_s}{L_v} \right) \]

In dimensional form, the burning rate is

\[ \dot{m} = \frac{4\pi a \lambda}{c_p} \ln \left(1 + \frac{c_p(T_\infty - T_B) + QY_{O\infty}/\nu_O W_O}{Q_v} \right) \]

\[ B_c = \frac{c_p(T_\infty - T_B) + QY_{O\infty}/\nu_O W_O}{Q_v} = \text{impetus for transfer} \]

\[ = \text{resistance to transfer} \]

\[ \text{transfer number} \]

The flame standoff distance

\[ \tilde{r}_f = \frac{a \ln(1 + B_c)}{\ln(1 + \nu_F W_F Y_{O\infty}/\nu_O W_O)} \]

Note that \( r_f > 1 \) implies that \( c_p(T_\infty - T_B) > (Q_s - Q/\nu_F W_F)\nu^{-1}Y_{O\infty} \) which is not restrictive because typically \( Q > Q_s \).

The flame temperature

\[ \tilde{T}_f = T_\infty + \frac{(Q/\nu_F W_F - Q_v)/c_p - (T_\infty - T_B)}{1 + \nu_O W_O/\nu_F W_F Y_{O\infty}} \]
If the activation energy parameter is considered large, the asymptotic analysis provides a more complete picture spanning the entire range of Damköhler number $D$.

Extinction may occur as the droplet size diminishes below a critical value; auto-ignition may occur if the ambient pressure is increased significantly.
Droplet history:
( use back the notation \( a = r_s \) and \( \rho = \rho_g \) )

The vaporization and burning rates were found to have the form 
\[ \dot{m} = 4\pi r_s(\lambda/c_p) \ln(1 + B) \]
with the appropriate transfer number. 

\[ \frac{dr_s}{dt} = -\frac{\rho_g v}{\rho_l} = \frac{k}{2r_s} \]

with 
\[ k = \frac{2\lambda \ln(1 + B)}{c_p \rho_l} \]

\[ 2r_s \frac{dr_s}{dt} = -k \]

\[ r_s^2 = r_s^2(0) - kt \]

d^2-law 
\[ d^2 = d^2(0) - Kt \]

\[ K = \frac{8(\lambda/c_p)}{\rho_l} \ln(1 + B) \]

\[ t_{life} = \left( \frac{\rho_l}{\rho_g} \right) \frac{d_0^2}{8D_{th} \ln(1 + B)} \]

The \( d^2 \)-law prediction is verified experimentally for most of the droplet lifetime with the exception, perhaps, of a short initial period following ignition.
The behavior of a single droplet in a combustion chamber depends on the Reynolds number $Re = \rho_d d_0 |u - u_p|/\mu_g$, where $u$ is a characteristic velocity of the gas and $u_p$ the particle velocity.

So far, the effect of the relative velocity of the ambient gas on the droplet has been neglected, practically taking $Re = 0$. The “spherical symmetry” assumption cannot be justified when $Re \neq 0$. A general approach that has been often used to address the problem of droplet combustion in a convective field has been to correct a-posteriori the results derived for $Re = 0$ with empirical correlations.

Formulas that have been used are of the form

$$K = K^{(0)} \left( 1 + C Re^{1/2} Pr^{1/3} \right) \quad C \approx 0.3$$

where $K$ is the evaporation constant, $K^{(0)}$ the evaporation constant for a stationary burning droplet, $Pr = \mu_g/D_{th}\rho_g$ is the Prandtl number.

Similarly, burning rate corrections for free convection has been proposed, in the form

$$K = K^{(0)} \left( 1 + C Gr^{0.52} \right) \quad C \approx 0.5$$

where $Gr$ is the Grashof number, representing the ratio of the buoyancy to viscous forces.

$$Gr = \frac{\rho_d^2 d_0^3 g}{\mu_g^2} \left( \frac{\Delta T}{T} \right)$$

Here $\Delta T$ may be taken as the difference between the flame temperature and the ambient temperature.

The assumption of spherical symmetry for suspended droplets can be justified either in micro-gravity conditions ($g \ll 1$) or in reduced pressure environment ($P_0 \sim \rho_g \ll 1$).
Combustion of a solid particle:

Similar to a fuel droplet a solid particle may be surrounded by a diffusion flame, so that

\[ \dot{m} = \frac{4\pi r_s \lambda}{c_p} \ln (1 + B) \]

if the flame is close enough to the surface, \( r_f \approx 1 \)

\[ \ln (1 + B) = \ln (1 + \nu^{-1}Y_O) \]

\[ \dot{m} = 4\pi r_s \rho g \Delta \ln (1 + \nu^{-1}Y_O) \]

This is the diffusion-controlled limit, where the diffusion of oxidizer towards the particle controls the burning.

For \( Le = 1 \), we have \( \lambda/c_p = \rho \Delta \).

Using

\[ \dot{m} = -\frac{d}{dt} \left( \frac{2}{3} \pi \rho_s r_s^3 \right) = -4\pi \rho_s r_s^2 \frac{dr_s}{dt} \]

\[ \dot{m} = 4\pi r_s \rho g \Delta \ln (1 + \nu^{-1}Y_O) \]

we find in the diffusion-controlled limit

\[ r_s^2 = r_s^2(0) - \frac{2\rho g \Delta}{\rho_s} \ln (1 + \nu^{-1}Y_O) \cdot t \]

\[ d^2 = d_0^2 - \frac{8\rho g \Delta}{\rho_s} \ln (1 + \nu^{-1}Y_O) \cdot t \]

namely a \( d^2 \)-law.

\[ t_{life} = \left( \frac{\rho_s}{\rho_g} \right) \frac{d_0^2}{8\Delta \ln(1 + \nu^{-1}Y_O)} \]
If burning occurs on the surface (and possibly inside the pores) of the particle, the burning rate is controlled by the chemical kinetics. For an Arrhenius-type heterogenous surface chemical reaction rate, we have

\[ \dot{m} = 4\pi r_s^2 k_s \rho Y_O \]

where \( k_s = B_s e^{-E/RT_s} \)

This is the kinetic-controlled limit, where the chemical reaction rate controls the burning.

Using

\[ \dot{m} = -\frac{d}{dt} \left( \frac{4}{3} \pi \rho_s r_s^3 \right) = -4\pi \rho_s r_s^2 \frac{dr_s}{dt} \]

\[ r_s = r_s(0) - \frac{k_s}{\rho_s} \rho Y_O \ t \]

\[ d = d_0 - 2k_s \rho Y_O \ t \]

namely a \( d^1 \)-law (linear).

\[ t_{life} = \left( \frac{\rho_s}{\rho_y} \right) \frac{d_0}{2B_s Y_O e^{-E/RT_s}} \]

The controlling mechanism is determined by a surface Damköhler number

\[ D_s = \frac{t_{diffusion}}{t_{kinetic}} = \frac{d_0 B_s Y_O e^{-E/RT_s}}{4D_s \ln(1+\nu^{-1}Y_O)} \]

large particles, or higher ambient temperature \( \Rightarrow D_s \) large
\[ \Rightarrow \text{diffusion-controlled} \]

small particles, or low ambient temperature \( \Rightarrow D_s \) small
\[ \Rightarrow \text{kinetic-controlled} \]
Spray Modeling:

The accurate description of spray combustion, say by DNS, requires identification of whether liquid or gas is present at every point in space, to employ appropriate equations of state in each of the separate phases and to determine the evolution of a large number of liquid-gas interfaces with boundary conditions that involve phase transformation. The large number of droplets required to simulate real dispersed flows in turbulent environments that are encountered in practical applications, make such calculations prohibitive.

There are a practically three approaches to spray combustion modeling.

- Continuum formulation
- Lagrangian formulation
- Probabilistic formulation

We will not be discussing any of these in detail, but we will try to get a general idea of what each of these formulations involve.

Continuum Formulation. Properties of both the particles and the gas are assumed to exist at every spatial point (the average value of a small parcel that includes both phases) regardless of whether that point is actually in the liquid, solid or gas at that instant.

The spray contains $N$ distinct chemical species and $M$ different class of droplets identified by fuel content, size and/or velocity. Let $m_i$ be the mass of species $i$ and $\hat{m}^{(k)}$ the mass of droplets of class $k$.

The total volume of the mixture will be denoted by $V_T$ and will be given by $V_T = V_g + V_l$, where $V_g$ and $V_l$ are the volumes occupied by the gas and the liquid phases, respectively.

$\varphi = V_g/V_T$ is the volume fraction of the gas, namely the ratio of the volume occupied by the gas to the total volume of the mixture.

$1 - \varphi$ is the void fraction, or the ratio of the volume occupied by the condensed phase from the total volume $V_T$. 
The bulk density (mass per total volume $V_T$) of species $i$ is $\rho_i$.

The bulk density of the gaseous mixture is $\rho = \frac{1}{V_T} \sum_{i=1}^{N} \rho_i$.

It differs from the gas density $\rho_g = \frac{1}{V_g} \sum_{i=1}^{N} m_i = \frac{V_T}{V_g} \rho = \frac{1}{\varphi} \rho$ which is the mass per unit volume that includes only gas.

Similarly, the density of droplets of class $k$ is $\hat{\rho}^{(k)}$, so that the bulk density of droplets is $\hat{\rho} = \frac{1}{V_T} \sum_{k=1}^{M} \hat{\rho}^{(k)}$.

which differs from the liquid density $\rho_l = \frac{1}{V_l} \sum_{k=1}^{M} \hat{m}^{(k)} = \frac{V_T}{V_l} \rho_l = \frac{1}{1 - \varphi} \rho_l$

For a dilute spray the condensed phase volume fraction is small $(1 - \varphi \ll 1)$ so that $\rho \sim \rho_g$, but $\hat{\rho} = (1 - \varphi) \rho_l \sim \frac{1}{\varphi} \frac{\rho_l}{\hat{\rho}_l}$, which is the mass per unit volume that includes only gas.

$\hat{\rho}$ is often referred to as the spray density.

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Statements of mass conservation of the gas and condensed phases lead to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \dot{M}$$

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla \cdot (\hat{\rho} \hat{\mathbf{v}}) = -\dot{M}$$

Here $\dot{M}$ is the total mass (per unit volume, per unit time) added to the gas phase due to vaporization and $\mathbf{v}, \hat{\mathbf{v}}$ are the gas and particle velocities.

This can be determined, for example, from the expressions we derived for the vaporization rate of isolated droplets.

Note: $\frac{\partial}{\partial t} (\rho + \hat{\rho}) + \nabla \cdot (\rho \mathbf{v} + \hat{\rho} \hat{\mathbf{v}}) = 0$ i.e., the total mass is conserved.

Using $\hat{\rho} = (1 - \varphi) \rho_l$, we can recast the mass conservation of the droplets in the following form

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{v}) = \frac{\dot{M}}{\rho_l} + \nabla \cdot \hat{\mathbf{v}}$$

$\varphi$ is no longer conserved; the volume occupied by the droplets changes as a result of (i) vaporization, and (ii) distance between droplets.
Similarly, separate statements of momentum and energy conservation are written for the gas and liquid phases with appropriate source/sink terms representing the total momentum and energy carried to the gas from the vaporizing droplets, the total force exerted on the particles from their surroundings and the internal heating of the droplets which, in general, could have a different temperature than the bulk gas.

**Probabilistic Formulation.** In this approach the condensed phase is described by a distribution function, or probability density function (*pdf*)

\[ f(r, x, v, e, t) \]

\[ f \, dr \, dx \, dv \, de \ldots \] is the probable number of particles in the radius range \( dr \) about \( r \), in the spatial range \( dx \) about the position \( x \), with velocities in the range \( dv \) about \( v \) and internal energy in the range \( e \) about \( de \), etc..., at time \( t \).

The time evolution of the distribution function \( f \) is given by

\[
\frac{\partial f}{\partial t} + \nabla_r \cdot (vf) + \nabla_v \cdot (af) + \frac{\partial}{\partial r}(rf) + \frac{\partial}{\partial e}(ef) + \cdots = Q,
\]

and is known as the spray equation.

The terms on the l.h.s. of the spray equation represent drop-carrier gas interactions, the source term \( Q \) on the r.h.s. represents particle-particle interactions (fragmentation, coalescence, etc.).
The quantities in the spray equation are determined by the laws governing the dynamics of individual droplets

\[ \frac{dx}{dt} = v, \quad \frac{dv}{dt} = a, \quad \frac{dr}{dt} = \dot{r}, \quad \frac{dc}{dt} = \dot{c}, \quad \text{etc...} \]

For example, the dependence of \( \dot{r} \) on the radius for liquid drops can be specified by adopting the \( d^2 \)-law

\[ \dot{r} = \frac{\lambda/c_p \ln (1 + B)}{r}, \]

The conservation laws for the gas-droplet mixture will include additional source terms involving the integral of \( f \). For example, the mass and momentum equations are

\[ \frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \mathbf{u}) = - \int 4\pi r^2 \rho \dot{r} f \, dr \, dv \, de, \]

\[ \rho_f (\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla P + \rho_f g - \int \left[ \frac{2}{3} \pi r^3 \rho \mathbf{a} + 4\pi r^2 \rho \dot{r} (\mathbf{v} - \mathbf{u}) \right] f \, dr \, dv \, de, \]

where \( \rho_f \) is the fluid density, namely the mass of the gas mixture per unit volume of space, which due to the presence of the particles differs from the gas density \( \rho_g \) that corresponds to the mass per unit volume occupied by the gas.

The main difficulty in this approach is the high dimensionality of the distribution function \( f(r, x, v, c, t) \) which, under the simplest assumptions, contains nine variables.

**Lagrangian Discrete Particle Formulation.** The droplets are considered as individual point-particles and their trajectories are solved for, separately from the flow field which exists only where there are no droplets. The main difficulty is probably the inability to track a sufficiently large number of particles in order to