Lectures on
Dynamics of Gaseous Combustion Waves
(from flames to detonations)

Professor Paul Clavin
Aix-Marseille Université
ECM & CNRS (IRPHE)

Lecture XIII
Stability analysis of shock waves
Lecture 13: **Stability analysis of shock waves**

13-1. Acoustic waves and entropy-vorticity wave

   *Linearized Euler equations*
   *Linearized flow field*

13-2. Analyses

   *Dispersion relation for general materials*
   *Classification of normal modes*
   *Spontaneous emission of sound and instability*
   *Stability of shocks in ideal gases*
   *Stability of reacting shocks*
XIII-1) Acoustic waves and entropy-vorticity wave

Shock wave $\approx$ hydrodynamic discontinuity + Rankine-Hugoniot conditions
The flow of shocked gas in a planar wave is uniform
\[ D > a_u \Rightarrow \text{the upstream flow in a wrinkled shock is not perturbed} \]
Flow velocity $\overline{u}_N$ is sufficiently large $\Rightarrow$ the diffusive fluxes are negligible: $Ds/Dt = 0$
The entropy of shocked gas is modified at the Neumann state of a wrinkled shock $\Rightarrow \nabla s(\mathbf{r}, t) \neq 0$

**Linearized Euler equations**
(written in 2-D for simplicity. Extension to 2-D is straightforward)
\[
\begin{align*}
    u &= \overline{u}_N + \delta u, \quad w = \delta w, \quad \rho = \overline{\rho}_N + \delta \rho, \quad p = \overline{p}_N + \delta p \\
    \frac{1}{\overline{\rho}_N} \frac{D}{Dt} \delta \rho + \frac{\partial}{\partial x} \delta u + \frac{\partial}{\partial y} \delta w &= 0, \\
    \frac{D}{Dt} = \partial / \partial t + \overline{u}_N \partial / \partial x \\
    \overline{\rho}_N \frac{D}{Dt} \delta u &= -\frac{\partial}{\partial x} \delta p, \quad \overline{\rho}_N \frac{D}{Dt} \delta w = -\frac{\partial}{\partial y} \delta p, \\
    \delta p / \delta \rho |_{s = c s t} &\equiv a \\
    \frac{D}{Dt} \delta s = 0 \Rightarrow \frac{D}{Dt} \delta p = \overline{a}_N^2 \frac{D}{Dt} \delta \rho,
\end{align*}
\]

*Wave equation for the pressure (d’Alembert equation)*

eliminating $\delta \rho$ $\Rightarrow$
\[
\frac{1}{\overline{\rho}_N \overline{a}_N^2} \frac{D}{Dt} \delta p + \frac{\partial}{\partial x} \delta u + \frac{\partial}{\partial y} \delta w = 0
\]

eliminating $\delta u$ and $\delta w$ $\Rightarrow$
\[
\frac{D^2}{Dt^2} \delta p - \overline{a}_N^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \delta p = 0
\]

the pressure fluctuations are fully propagated by acoustic waves in the shocked gas moving at constant velocity $\overline{u}_N$
### Linearized flow field

#### Flow splitting

**Acoustic wave + vorticity wave**

\[
\delta p = \delta p^{(a)}, \quad \delta u = \delta u^{(a)} + \delta u^{(i)} \quad \delta w = \delta w^{(a)} + \delta w^{(i)}
\]

\[
\frac{\partial}{\partial t} \delta u^{(i)} + \frac{\partial}{\partial x} \delta u + \frac{\partial}{\partial y} \delta w = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t} \delta u^{(i)} + \frac{\partial}{\partial x} \delta u^{(i)} + \frac{\partial}{\partial y} \delta w^{(i)} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t} \delta u^{(i)} + \frac{\partial}{\partial x} \delta u^{(i)} = \overline{\nu}_N \frac{\partial}{\partial y} \delta w^{(i)}
\]

**Normal-mode analysis**

\[
\alpha(y, t) = \hat{\alpha} e^{iky + \sigma t}
\]

\[
\delta p = \hat{\tilde{p}}_N \exp (i\pm x + iky + \sigma t)
\]

\[
x = 0 : \quad \delta p = \delta p_N(y, t) = \hat{\tilde{p}}_N e^{iky + \sigma t}
\]

\[
\delta f(x, y, t) = \hat{\tilde{f}}(x)e^{iky + \sigma t}
\]

\[
\sigma \in \mathbb{Z} \quad (\sigma(k))
\]

#### Incompressibility condition

\[
\frac{\partial}{\partial t} \delta u^{(i)} = \pi_N \frac{\partial}{\partial y} \delta w^{(i)}
\]

\[
-\frac{\partial}{\partial t} \delta u^{(i)} - \frac{\partial}{\partial y} \delta w^{(i)} = \frac{(i\pm + \overline{\nu}_N k^2)}{\overline{\nu}_N + \sigma} \overline{\nu}_N \overline{\nu}_N - \sigma \overline{\nu}_N + i k \overline{\nu}_N = 0
\]
XIII-2 Analyses

Dispersion relation for general materials

Compatibility condition

\[
\left( \frac{\partial}{\partial t} + \tilde{u}_N \frac{\partial}{\partial x} \right)^2 \delta p - \tilde{p}_N \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \delta p = 0
\]

\[
\left( \sigma + il_{\pm} \bar{u}_N \right)^2 + \bar{u}_N^2 \left( l_{\pm}^2 + k^2 \right) = 0 \Rightarrow \sigma^2 + 2il_{\pm} \bar{u}_N - l_{\pm}^2 \bar{u}_N^2 + \bar{u}_N^2 \left( l_{\pm}^2 + k^2 \right) = 0
\]

\[
-\left( \tilde{p}_N / \bar{p}_N \bar{u}_N \right) - \frac{\tilde{u}_N}{\bar{u}_N} + ik \tilde{w}_N = 0 \Rightarrow \sqrt{1 - M_N^2} \left[ \pm \sqrt{1 + S^2} \right] \frac{\tilde{p}_N}{\bar{p}_N \bar{u}_N} - \frac{\tilde{u}_N}{\bar{u}_N} + ik \tilde{w}_N = 0
\]

\[
\pm \sqrt{S^2 + 1} \frac{\tilde{p}_N}{\bar{p}_N \bar{u}_N} + S \frac{\tilde{u}_N}{\bar{u}_N} - \frac{\tilde{w}_N}{\bar{u}_N} \frac{M_N}{\sqrt{1 - M_N^2}} = 0
\]

\( (\text{Buckmaster Ludford } 1988, \text{ Clavin et al. } 1997) \)

The Rankine Hugoniot relations yields an equation for \( S \propto \frac{\sigma}{\bar{u}_N \left| k \right|} \)

\[
\frac{\delta p_N}{\bar{p}_N} \propto \tilde{\alpha} \frac{\tilde{u}_N}{\bar{u}_N}, \quad \frac{\tilde{p}_N}{\bar{p}_N} \propto i\sigma \frac{\tilde{\alpha}}{\bar{u}_N}, \quad \frac{\delta u_N}{\bar{u}_N} \propto \tilde{\alpha} \frac{\tilde{u}_N}{\bar{u}_N}, \quad \tilde{u}_N \propto i \sigma \frac{\tilde{\alpha}}{\bar{u}_N}, \quad \frac{\delta w_N}{\bar{u}_N} \propto \alpha' \frac{\tilde{w}_N}{\bar{u}_N} \propto ik \tilde{\alpha}
\]

Downstream boundary condition

\( x \to \infty \) : bounded condition (in the unstable case, \( \text{Re} \sigma > 0 \))

\[
\delta p = \tilde{p}_N \exp \left( il_{\pm} x + iky + \sigma t \right)
\]

selection of the the sign in \( l_{\pm} \) such that \( e^{il_{\pm} x} \) does not diverge
**Rankine Hugoniot relations (general material)**

\[
P = \frac{D}{a}
\]

mass
\[
\rho_N(u_N - \partial \alpha / \partial t - w_N \partial \alpha / \partial y) = \rho_u(\frac{D}{a} - \partial \alpha / \partial t)
\]

\[\Rightarrow \delta \rho_N \bar{u}_N + \bar{u}_N(\delta u_N - \partial \alpha / \partial t) = -\rho_u \partial \alpha / \partial t, \quad \delta m = -\rho_u \partial \alpha / \partial t\]

\[
tangential \quad \text{momentum}
\]
\[
w_N = (\bar{D} - u_N)\alpha'\n\]
\[\Rightarrow \delta w_N = (\bar{D} - \bar{u}_N)\partial \alpha / \partial y\]

\[
longitudinal \quad \text{momentum}
\]
\[
p_N - p_u = -m^2 \left( \frac{1}{\bar{p}_N} - \frac{1}{\bar{u}_N} \right)
\]
\[\Rightarrow \delta p_N \bar{p}_N = 2 \left( \frac{1}{\bar{p}_N} - \frac{1}{\bar{u}_N} \right) \rho_u \partial \alpha / \partial t + \frac{m^2}{\bar{p}_N \bar{u}_N} \delta \rho_N
\]
\[\Rightarrow \delta \rho_N \bar{p}_N = -2 \left( \frac{1}{\bar{p}_N} - \frac{1}{\bar{u}_N} \right) \rho_u \partial \alpha / \partial t, \quad \delta \rho_N \bar{p}_N = -2 \left( \frac{p_u - 1}{\bar{u}_N} \right) \frac{r}{1 - r} \frac{\partial \alpha / \partial t}{\bar{D}}
\]
\[\delta u_N = \left( \frac{p_u - 1}{\bar{u}_N} \right) \frac{1 + r}{1 - r} \frac{\partial \alpha / \partial t}{\bar{D}}, \quad \delta w_N = \left( \frac{p_u - 1}{\bar{u}_N} \right) \frac{\partial \alpha / \partial y}{\bar{D}}
\]

**Linear rate**

\[\pm \sqrt{S^2 + 1} \frac{\bar{u}_N - \bar{D}}{\bar{u}_N \bar{p}_N \bar{u}_N} = -S \frac{\bar{u}_N}{\bar{p}_N} + \frac{ik \bar{u}_N}{\bar{p}_N \bar{u}_N} \frac{\bar{M}_N}{\sqrt{1 - \bar{M}_N^2}} \sqrt{1 + \bar{M}_N^2}
\]

\[\alpha(y,t) = \hat{\alpha} e^{iky + \sigma t}, \quad \frac{\partial \alpha}{\partial t} = \sigma \alpha, \quad \frac{\partial \alpha}{\partial y} = ik \alpha
\]

\[\Rightarrow \pm 2 \bar{M}_N S \sqrt{1 + S^2} = (1 + r) S^2 + (1 - r)n
\]

\[S = \frac{\sigma}{\bar{u}_N |k|} \sqrt{1 - \bar{M}_N^2}
\]

\[\sigma^2 \bar{M}_N^2 k^2 = \frac{1}{1 - \bar{M}_N^2}
\]

\[a \equiv (1 + r)^2 - 4 \bar{M}_N^2, \quad b \equiv (1 - r^2)n - 2 \bar{M}_N^2, \quad c \equiv (1 - r^2)n^2 > 0.
\]

**Quadratic equation for** \[\sigma^2 / \bar{a}_N^2 k^2\]

Clavin Williams 2012
Classification of normal modes

Normal modes for general materials

\[ \pm 2M_N S \sqrt{1 + S^2} = (1 + r) S^2 + (1 - r) n \]

\[ a S^4 + 2b S^2 + c = 0 \]

\[ S = \frac{\sigma}{\alpha_N |k|} \frac{1}{\sqrt{1 - M_N^2}} \]

\[ a = (1 + r)^2 - 4M_N^2, \quad b = (1 - r^2)n - 2M_N^2, \quad c = (1 - r^2)n^2 > 0. \]

only the roots that satisfy boundedness condition should be retained  \( \text{Re}(i \pm) \leq 0, \quad \text{Re} \left( \frac{\tilde{M}_N S \pm \sqrt{1 + S^2}}{1 + \tilde{M}_N^2} \right) \leq 0 \)

\( \text{Re}(\sigma) < 0 : \) stable mode exponentially damped

\( \text{Re}(\sigma) > 0 : \) unstable mode exponentially amplified

\( S^2 < 0 : \) \( \text{Re}(\sigma) = 0, \quad \omega \equiv \text{Im}(\sigma) \neq 0 \) neutral oscillatory modes \( S = \pm i \Omega, \quad \Omega \sqrt{1 - M_N^2} \equiv \omega / (\pi_N |k|) > 0 \quad \Omega > 1 \quad i l_{\pm} = \pm i l, \)

\[ \delta p = \tilde{p}_N \exp [i K, \quad (r - \pi_N e_x + \pi_N e_K t)] \quad \omega = \pi_N \sqrt{l^2 + k^2 - \pi_N l} \quad (l / |k|) \sqrt{1 - M_N^2} = \tilde{M}_N \Omega + \sqrt{\Omega^2 - 1} > 0 \]

longitudinal component of the velocity (unperturbed shock) of the sound wave: \( e_x \cdot (\overline{u}_N e_x - \overline{u}_N e_K) = \frac{\overline{u}_N - \overline{u}_N}{\sqrt{l^2 + k^2}} \)

Neutral oscillatory modes

Spontaneous generation of sound. Radiating condition: \( \overline{u}_N \sqrt{l^2 + k^2 - \overline{u}_N l} > 0 \)

Non-radiating condition: \( \overline{u}_N \sqrt{l^2 + k^2 - \overline{u}_N l} < 0 \)

Classification of the normal modes in the parameters space

\[ r_s = n - \sqrt{(1 - M_N^2) \left( 1 - \frac{\overline{u}_N}{\tilde{u}_N} \right)} \]

\[ \frac{n}{n + 1} \]

Clavin Williams 2012
Spontaneous emission of sound and instability

D’Yakov Kontorovich 1954-57

Oscillatory neutral modes

\[ \pm 2M_N S \sqrt{1 + S^2} = (1 + r)S^2 + (1 - r)n \]

neutral oscillatory mode

\[ S = i\Omega, \quad \Omega > 1 \]

\[ \Rightarrow \begin{cases} \text{radiating waves: } \frac{l}{|k|} = \left[ M_N\Omega - \sqrt{\Omega^2 - 1} \right] / \sqrt{1 - M_N^2}, / \sqrt{1 - M_N^2}, \\ 2M_N\Omega - \sqrt{\Omega^2 - 1} = -\Omega^2 (1 + r) + (1 - r)n > 0, \\ -2M_N\Omega + \sqrt{\Omega^2 - 1} = -\Omega^2 (1 + r) + (1 - r)n < 0 \end{cases} \]

Non-radiating waves:

\[ \frac{l}{|k|} = \left[ M_N\Omega + \sqrt{\Omega^2 - 1} \right] / \sqrt{1 - M_N^2}, \]

\[ 2M_N\Omega + \sqrt{\Omega^2 - 1} = -\Omega^2 (1 + r) + (1 - r)n > 0, \]

\[ -2M_N\Omega - \sqrt{\Omega^2 - 1} = -\Omega^2 (1 + r) + (1 - r)n < 0 \]

Transmitted or reflected sound wave

\[ D > a_u \quad \Rightarrow \quad u_N < a_N \]

shocked material

\[ \begin{array}{c|c|c} \text{incident} & \text{transmitted} & \text{reflected} \\ \hline D > a_u & u_N < a_N & \hline \end{array} \]

If a normal mode is radiating the response of the shock diverges when the reflected (or transmitted) waves matched the radiating normal mode

\[ \frac{\hat{p}_t}{\hat{p}_i} = \left[ \frac{2M_N\Omega + \sqrt{\Omega^2 - 1} - \Omega^2 (1 + r) + (1 - r)n}{-2M_N\Omega - \sqrt{\Omega^2 - 1} - \Omega^2 (1 + r) + (1 - r)n} \right] \quad \text{denominator goes through 0} \]

A neutral oscillatory mode that is radiating is considered as unstable

D’Yakov Kontorovich 1954-57

Power laws of neutral modes

Damping \((n < 0)\) or amplification \((n > 0)\) involving power laws \(t^n\)

Neutral modes with non-radiating acoustic waves relax following a power law in time \(t^n, \ n < 0\)

Neutral modes with a radiating acoustic wave is unstable according a power law in time \(t^n, \ n > 0\)

\[ \begin{array}{c|c|c} \text{UNSTABLE} & \text{STABLE} \\ \text{oscillatory modes} & \text{oscillatory modes} \\ \text{Re}(\sigma) > 0 & \text{Re}(\sigma) = 0 & \text{Re}(\sigma) < 0 \\ \text{radiating} & \text{non-radiating} & \text{radiating} \\ \sigma = \frac{-1 + 2M_N}{r} & 0 & \frac{-1 + 2M_N}{r} \\ -1 & 1 & \frac{-1 + 2M_N}{r} \\ 1 & \frac{-1 + 2M_N}{r} & \end{array} \]

UPS 2007

Dunlop 1964
Stability of shocks in ideal gases

polytropic gas, $\gamma = \text{cst.}$

\[
\begin{align*}
\frac{u_N}{\rho} &= \frac{\gamma}{(\gamma + 1)M_u^2 + 2}, \\
\frac{\rho N}{\rho u} &= \frac{2\gamma M_u^2 - (\gamma - 1)}{(\gamma + 1)}, \\
M_u^2 &= \frac{(\gamma - 1)M_u^2 + 2}{2\gamma M_u^2 - (\gamma - 1)}
\end{align*}
\]

\[
r \equiv -\frac{(\rho uD)^2}{d\rho N/d\rho u} = \frac{1}{M_u^2}, \\
n \equiv \frac{\rho N}{\rho u} \left( \frac{M_u^2}{M_u^2 - 1} \right) = \frac{M_u^2}{M_u^2 - 1}
\]

\[
\pm 2M_N S \sqrt{1 + S^2} = (1 + r)S^2 + (1 - r)n, \quad \Rightarrow \quad \pm 2S M_N \sqrt{1 + S^2} = S^2 (1 + M_u^{-2}) + 1
\]

\[
\begin{align*}
M_u > 1, \quad \gamma > 1 \quad \Rightarrow \quad (n - 1)/(n + 1) < r < r^*
\end{align*}
\]

\[
r^* = \frac{n - \sqrt{(1 - M_u^2) (1 - \frac{n}{m})}}{n + 1}, \quad \frac{1}{2M_u^2 - 1} < \frac{1}{M_u^2} < \frac{M_u^2 - (M_u^2 - 1)^2 \sqrt{2M_u^2[2\gamma M_u^2 - (\gamma - 1)] - 1}}{2M_u^2 - 1}
\]

Shock waves in polytropic gases have neutral modes with non-radiating acoustic waves

They are stable with a relaxation of initial disturbances in power laws $1/t^{3/2}$

---

OK with experiments
Freeman 1957 Lapworth 1959

Formation of Mach stems (see next lecture)
Stability of reacting shocks

Clavin Williams 2009 2013

Reacting shocks = detonations considered as an hydrodynamic discontinuity

thickness = 0: no modification of the inner structure

\[ r \equiv -\frac{\rho_u D^2}{dp_b/\rho_b} = \frac{M_u^2}{1 - \frac{V_b}{M_u^2}} \left[ 1 + \frac{V_b}{M_u^2} \right] \]

\[ \chi \equiv \sqrt{\left(1 - \frac{M_u^2}{\chi_m^2}\right)^2 - 4\chi M_u^2} \]

\[ r = \frac{(1 - \chi) + \frac{1}{M_u^2}}{(1 + \chi) + \frac{1}{M_u^2}} = \left(\frac{n - 1}{n + 1}\right) \left[ 1 + \frac{1}{M_u^2(1 - \chi)} \right] \]

\[ \left(\frac{n - 1}{n + 1}\right) \leq r \]

Overdriven reacting shocks in polytropic gases have neutral modes with non-radiating acoustic waves

They are stable with a relaxation of initial disturbances in power laws

For the CJ marginal regime the acoustic waves in the burned gas propagate in the direction parallel to the unperturbed planar solution

Clavin Williams 2009 ≠ Majda Rosales 1983