Lectures on
Dynamics of Gaseous Combustion Waves
(from flames to detonations)

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Lecture IV
Hydrodynamic instability of flames
Lecture 4: **Hydrodynamic instability of flames**

4-1. Jump across an hydrodynamic discontinuity
4-2. Linearized Euler equations of an incompressible fluid
4-3. Conditions at the front
4-4. Dynamics of passive interfaces
4-5. Darrieus-Landau instability
4-6. Curvature effect: a simplified approach
IV - 1) **Jump across an hydrodynamic discontinuity**

\[ \Lambda \gg d_L \]

flame considered as a discontinuity
flame thickness and curvature neglected

flame \( \approx \) surface of zero thickness separating two incompressible flows

Low Mach nb approx + inviscid approx: Euler eqs

\[
\begin{align*}
\partial \rho / \partial t &= -\nabla \cdot (\rho \mathbf{u}) \\
\rho \partial \mathbf{u} / \partial t &= -\nabla p \\
&\iff \partial (\rho \mathbf{u}) / \partial t = -\nabla \left( \rho \mathbf{l} + \rho \mathbf{u} \mathbf{u} \right)
\end{align*}
\]

tilted planar front

reference frame of the flame \( \mathbf{r} = (x, z), \mathbf{u} = (u, w) \)

\[
\begin{align*}
\partial \rho / \partial t &= -\partial (\rho u) / \partial x - \partial (\rho w) / \partial z, \\
\partial (\rho u) / \partial t &= -\partial (p + \rho u^2) / \partial x - \partial (\rho uw) / \partial z, \\
\partial (\rho w) / \partial t &= -\partial (\rho uw) / \partial x - \partial (p + \rho w^2) / \partial z
\end{align*}
\]

jump relations (reference frame of the flame)

\[
\begin{align*}
[\rho u]_+^- &= 0 \\
[p + \rho u^2]_+^- &= 0 \\
\rho u \neq 0 \Rightarrow [w]_+^- &= 0
\end{align*}
\]

\[
\lim_{d_L \to 0} \int_{d_L}^a(x, y, t)dx = 0
\]

if \( a(\mathbf{r}, t) \) is regular
reference frame of the flame front

\[
\frac{U_b}{U_L} = \frac{\rho_u}{\rho_b} = \frac{T_b}{T_u}
\]

"instantaneous" modification of the flow field, both upstream and downstream
(low Mach nb approx: the speed of sound is infinite, \( a \approx \infty \))

tilted front

deviation of the stream lines

\[
\sigma \propto \frac{U_L}{\Lambda}
\]

\( \Lambda \gg d_L, \quad d_L/\Lambda \rightarrow 0 \)
equation of the perturbed front $x = \alpha(y, t)$ (reference frame $\mathcal{F}_0$)

flow velocity at the front $\mathbf{u}_f = (u_f, w_f)$

$$n_f = \left( \frac{1}{\sqrt{1 + \alpha_y^2}}, -\frac{\alpha_y'}{\sqrt{1 + \alpha_y^2}} \right), \quad u_n = \mathbf{u}_f \cdot n_f = (u_f - \alpha_y' w_f) / \sqrt{1 + \alpha_y^2},$$

$w_{tg} = (w_f + \alpha_y u_f) / \sqrt{1 + \alpha_y^2}$

normal velocity of the front $\mathbf{D}_f = \frac{\mathbf{\alpha}_t}{\sqrt{1 + \alpha_y^2}}$

flow velocity relative to the perturbed front $U_n = u_n - \mathbf{D}_f = (u_f - \mathbf{\alpha}_t - \alpha_y' w_f) / \sqrt{1 + \alpha_y^2}$

$W_{tg} = w_{tg}$

normal component

tangentail component

conservation of mass

$p^- U_n^- = p^+ U_n^+$

$p^- (u_f^- - \mathbf{\alpha}_t - \alpha_y' w_f^-) = p^+ (u_f^+ - \mathbf{\alpha}_t - \alpha_y' w_f^+)$

conservation of momentum

$[p + p U_n^2]^+ = 0 \quad [W_{tg}]^+ = 0$

$p_f^- + p - \frac{(u_f^- - \mathbf{\alpha}_t - \alpha_y' w_f^-)^2}{1 + \alpha_y^2} = p_f^+ + p + \frac{(u_f^+ - \mathbf{\alpha}_t - \alpha_y' w_f^+)^2}{1 + \alpha_y^2}$

$(w_f^- + \alpha_y u_f^-) = (w_f^+ + \alpha_y u_f^+)$
IV - 2) Linearised Euler equations of an incompressible fluid

\[ a = \bar{a} + \delta a \]
\[ \bar{m}_f = \bar{p}^- \bar{u}_f = \bar{p}^+ \bar{u}_f^+ \]
\[ \rho^+ \frac{\partial}{\partial t} + \bar{m}_f \frac{\partial}{\partial x} \delta u^\pm = -\frac{\partial}{\partial x} \delta \pi^\pm, \]
\[ \rho^- \frac{\partial}{\partial t} + \bar{m}_f \frac{\partial}{\partial x} \delta w^\pm = -\frac{\partial}{\partial y} \delta \pi^\pm, \]

\[ x \to +\infty : \text{disturbances remain finite,} \]
\[ x \to -\infty : \text{no disturbances,} \quad \delta u^- = 0 \]

\[ \delta a(x, y, t) = \tilde{a}(x, t)e^{ik \cdot y} \quad \alpha(y, t) = \tilde{\alpha}(t)e^{ik \cdot y} \]

pressure
\[ \frac{\partial^2 \tilde{\pi}^\pm}{\partial x^2} - |k|^2 \tilde{\pi}^\pm = 0 \]

flow velocity
\[ \frac{\partial \tilde{u}^\pm}{\partial x} + ik \tilde{w}^\pm = 0 \]
\[ \rho^\pm \left( \frac{\partial}{\partial t} + \tilde{u}^\pm \frac{\partial}{\partial x} \right) \tilde{u}^\pm(x, t) = \pm |k| \tilde{\pi}^\pm(t)e^{\mp |k| x} \]

general solution to the homogeneous equation + particular solution
\[ \tilde{u}^\pm(x, t) = \tilde{u}_R^\pm(x, t) + \tilde{u}_P^\pm(x, t) \]

\[ \frac{\partial \tilde{u}_R^\pm}{\partial t} + \tilde{u}^\pm \frac{\partial \tilde{u}_R^\pm}{\partial x} = 0, \]
\[ \rho^\pm \left( \frac{d}{dt} + \tilde{u}^\pm k \right) \tilde{u}_P^\pm(t) = \pm k \tilde{\pi}^\pm_f(t) \]
\[ \tilde{u}^\pm(x, t) = \tilde{u}_R^\pm(x, t) + \tilde{u}_P^\pm(x, t) \]

\[
k \equiv |k| = 2\pi/\Lambda \quad \tilde{u}^- = \tilde{u}_P^- = \tilde{u}_f^-(t)e^{kx} \quad \tilde{u}_P^+ = \tilde{u}_p^+(t)e^{-kx}
\]

\[
\frac{\partial \tilde{u}_R^\pm}{\partial t} + \bar{u}^\pm \frac{\partial \tilde{u}_R^\pm}{\partial x} = 0, \quad \tilde{u}_R^\pm = \tilde{u}_R^\pm(t - x/\bar{u}^\pm), \quad \tilde{u}_R^- = 0, \quad \tilde{u}_R^+ = \tilde{u}_R^+(t - x/\bar{u}^\pm),
\]

3 unknown functions: \( \tilde{u}_f^-(t), \tilde{u}_p^+(t), \tilde{u}_r^+(t) \)

\[
x < 0 : \quad \begin{cases} \tilde{u}^-(x, t) = \tilde{u}_f^-(t)e^{kx}, \\
k\tilde{\pi}^-(x, t) = -\tilde{\rho}^- \left( \frac{d}{dt} + \bar{u}^- k \right) \tilde{u}_f^-(t)e^{kx}, \end{cases}
\]

\[
x > 0 : \quad \begin{cases} \tilde{u}^+(x, t) = \tilde{u}_p^+(t)e^{-kx} + \tilde{u}_r^+(t - x/\bar{u}^+), \\
k\tilde{\pi}^+(x, t) = \tilde{\rho}^+ \left( \frac{d}{dt} - \bar{u}^+ k \right) \tilde{u}_p^+(t)e^{-kx}, \end{cases}
\]

\[
i k \dot{\tilde{w}}^-(x, t) = -k \tilde{u}^-(x, t), \quad i k \dot{\tilde{w}}^+(x, t) = -\frac{\partial}{\partial x} \tilde{u}^+(x, t).
\]

4 boundary conditions at the flame front involving the additional unknown \( \tilde{\alpha}(t) \)

2 for the conservation of mass (inner flame structure not modified)

\[
\delta m_f^- = \delta m_f^+ = 0 \quad m \equiv \rho(u - \partial \alpha/\partial t)
\]

2 for the conservation of normal and tangential momentum
IV-3) Conditions at the front

Mass

\[ \rho^- (\delta u_f^- - \dot{\alpha}_t) = \rho^+ (\delta u_f^+ - \dot{\alpha}_t) = 0 \]

\[ \Rightarrow \delta u_f^- = \delta u_f^+ = \dot{\alpha}_t \]

\[ \dot{u}_f^- (t) = \dot{u}_p^+ (t) + \dot{u}_f^+ (t) = d\ddot{\alpha}/dt \]

Tangential momentum

\[ \frac{\partial}{\partial y} (w_f^- + \alpha'_y \bar{u}^-) = \frac{\partial}{\partial y} (w_f^+ + \alpha'_y \bar{u}^+) \]

\[ \Rightarrow k\ddot{u}_p^+ (t) + \frac{1}{u^+} \frac{d\ddot{u}_p^+ (t)}{dt} + k\dddot{u}_f^- (t) = m_f \left( \frac{1}{\rho^+} - \frac{1}{\rho^-} \right) k^2 \ddot{\alpha} (t) \]

Normal momentum

\[ \delta p_f^- + 2\rho^- u_f^- (\delta u_f^- - \dot{\alpha}_t) = \delta p_f^+ + 2\rho^+ u_f^+ (\delta u_f^+ - \dot{\alpha}_t) \]

\[ x < 0: \begin{cases} \bar{u}^- (x, t) = \bar{u}_f^- (t)e^{kx}, \\ k\bar{u}^- (x, t) = -\rho^- \left( \frac{d}{dt} + \bar{u}^- \right) \bar{u}_f^- (t)e^{kx}, \end{cases} \]

\[ x > 0: \begin{cases} \bar{u}^+ (x, t) = \bar{u}_f^+ (t)e^{-kx} + \bar{u}_f^+ (t-x/\bar{u}^+), \\ k\bar{u}^+ (x, t) = \rho^+ \left( \frac{d}{dt} - \bar{u}^+ \right) \bar{u}_p^+ (t)e^{-kx}, \end{cases} \]

\[ \Rightarrow \bar{\pi}_f^- - \bar{\pi}_f^+ = (\rho^- - \rho^+) g(t) \ddot{\alpha} (t) \]

Equation for the front

\[ \ddot{u}_f^- = d\ddot{\alpha}/dt \Rightarrow \left( \rho^- + \rho^+ \right) \frac{d^2\ddot{\alpha}}{dt^2} + 2m_f k \frac{d\ddot{\alpha}}{dt} - k[(\rho^- - \rho^+) g(t) + (\bar{u}^+ - \bar{u}^-) m_f k] \ddot{\alpha} = 0 \]
IV-4) Dynamics of a passive interface

\[ \tilde{m}_f = 0 \]

\[
(\bar{\rho}^- + \bar{\rho}^+) \frac{d^2 \tilde{\alpha}}{dt^2} + 2 \bar{m}_f k \frac{d\tilde{\alpha}}{dt} - k [(\bar{\rho}^- - \bar{\rho}^+)g(t) + (\bar{u}^+ - \bar{u}^-) \tilde{u}_t k] \tilde{\alpha} = 0
\]

Fourier mode \[ \tilde{\alpha}(t) = \hat{\alpha} e^{\sigma t} \]
\[ \alpha(y, t) = \hat{\alpha}(t)e^{ik.y} \]

Rayleigh-Taylor instability

\[ g = \text{cst.} \quad (\bar{\rho}^- - \bar{\rho}^+)g > 0 \]

\[ \sigma^2 - A_t kg = 0, \quad A_t > 0 \]

Rayleigh-Taylor bubble (upwards propagation)

\[ g > 0, \quad A_t \equiv \frac{\rho^- - \rho^+}{\rho^- + \rho^+} > 0 \]
\[ \sigma = \sqrt{A_t kg} \]
\[ U_{\text{bubble}} = 0.361 \sqrt{2gR} \]

Gravity waves

\[ g = \text{cst.} \quad (\bar{\rho}^- - \bar{\rho}^+)g < 0 \]

\[ \tilde{\alpha}(t) = \hat{\alpha} e^{i\varpi t} \]
\[ \varpi \equiv \text{Im} \sigma \neq 0 \]
\[ \varpi = B \sqrt{gk} \quad B \equiv \sqrt{\frac{(\rho^+ - \rho^-)}{(\rho^+ + \rho^-)}} \]

Faraday (parametric) instability. Mathieu’s equation

\[ g(t) \text{ oscillating} \]
\[ \frac{d^2 \tilde{\alpha}}{dt^2} - \omega_o^2 [1 + \epsilon \cos(\varpi \tau)] \tilde{\alpha} = 0 \]
IV-5) Darrieus-Landau instability of flames

\[ g = 0 \quad \bar{m}_f = \bar{\rho}^- \bar{u}^- = \bar{\rho}^+ \bar{u}^+ \]

\[ (\bar{\rho}^- + \bar{\rho}^+ \frac{d^2 \bar{\alpha}}{dt^2} + 2\bar{m}_f k \frac{d\bar{\alpha}}{dt} - (\bar{u}^+ - \bar{u}^-)\bar{m}_f k^2 \bar{\alpha} = 0 \]

\[ \nu_b \equiv \bar{\rho}^- / \bar{\rho}^+ = \bar{u}^+ / \bar{u}^- > 1 \]

\[ \bar{u}_- \equiv U_L \quad \frac{\sigma}{U_L k} = \frac{1}{1 + \nu_b} \left[ -1 \pm \sqrt{1 + \nu_b - \nu_b^{-1}} \right] \]

\[ \sigma = AU_L k, \quad A > 0 \]

\[ d_L / \Lambda \to 0 : \text{no length scale in the problem; dimensional analysis \Rightarrow } \sigma \propto U_L k \]

\[ \rho_u \gg \rho_b : \sigma = \sqrt{U_b U_L k} \]

\[ (\rho_u - \rho_b) / \rho_u \ll 1 : \sigma = (U_b - U_L) k / 2 \]

\[ k = 2\pi / \Lambda \quad \text{shorter is the wavelength stronger is the instability !?} \]

\[ \text{however the analysis is valid only in the limit } d_L / \Lambda \to 0 \]

Stabilisation at small wavelength, \( \Lambda \approx d_L \)

\[ \frac{\partial \alpha}{\partial t} = BD_T \frac{\partial^2 \alpha}{\partial y^2} \quad \sigma_{\text{diff}} \equiv 1 / \tau_{\text{diff}} = -BD_T k^2 = -BU_L k (d_L k) \]

first order correction \( B > 0 ? \)

\[ kd_L < 1 : \quad \sigma = AU_L k - Bk^2 d_L + \ldots \]
VI-6) Curvature effect: a simplified approach

modification to the inner flame structure \( \delta m_f^- = \delta m_f^+ \neq 0 \)

first order in perturbation analysis \( d_L/\Lambda \ll 1 \)

\[
\delta m_f^- / \bar{\rho}^- \equiv (\delta u_f^- - \dot{\alpha}) = -BD_T \partial^2 \alpha / \partial y^2
\]

\[
\bar{m}_f(t) / \bar{m}_f \approx B d_L k^2 \bar{\alpha}(t)
\]

\( D_T = U_L d_L \)

Normal momentum

\[
\dot{\bar{p}}_f^- + 2\bar{\rho}_f^- (\delta u_f^- - \dot{\alpha}) = \dot{\bar{p}}_f^+ + 2\bar{\rho}_f^+(\delta u_f^+ - \dot{\alpha})
\]

(flame notations: \( p^+ \rightarrow \rho_b, \, p^- \rightarrow \rho_u, \, \rho_u > \rho_b \))

\[
\tilde{\bar{\pi}}_{f+} - \tilde{\bar{\pi}}_{f-} = -2\bar{m}_f \left( \frac{1}{\rho_b} - \frac{1}{\rho_u} \right) \dot{m}_f(t) + (\rho_u - \rho_b)g(t)\bar{\alpha}(t)
\]

equation for the flame front (correction due to curvature, finite thickness effect \( k d_L \neq 0 \))

\[
(r_u + \rho_b) \frac{d^2\tilde{\alpha}}{dt^2} + 2\bar{m}_f k \frac{d}{dt} \tilde{\alpha} (1 + B k d_L) = k \tilde{\alpha}(\rho_u - \rho_b) [g(t) + U_b U_L k (1 - 2B k d_L)]
\]

flame propagating downwards \( g < 0 \)

\[
\frac{1}{k_m} \equiv 2B d_L
\]

\[
\left( 1 + \frac{\rho_b}{\rho_u} \right) \frac{d^2\tilde{\alpha}}{dt^2} + 2U_L k \frac{d}{dt} \tilde{\alpha} = \left( \frac{\rho_u}{\rho_b} - 1 \right) k \left[ -\frac{\rho_b}{\rho_u} |g| + U_L^2 k \left( 1 - \frac{k}{k_m} \right) \right] \tilde{\alpha}
\]

non-dimensional parameters

\[
\nu_b \equiv \bar{\rho}^- / \bar{\rho}^+ = \bar{u}^+ / \bar{u}^- > 1
\]

\[
s = \sigma \tau_L, \quad \kappa \equiv kd_L, \quad \kappa_m \equiv 1 / (2B), \quad \mathcal{G}_0 \equiv (\rho_b / \rho_u) \text{Fr}^{-1}, \quad \text{Fr}^{-1} \equiv |g| d_L / U_L^2
\]

\[
(1 + \nu_b^{-1}) s^2 + 2\kappa s - (\nu_b - 1) \kappa \left[ -\mathcal{G}_0 + \kappa \left( 1 - \frac{\kappa}{\kappa_m} \right) \right] = 0
\]

Stability limits of flames propagating downwards \( \sigma = 0 \)

marginal wavenumber

\[
\left[ -\mathcal{G}_0 + \kappa \left( 1 - \frac{\kappa}{\kappa_m} \right) \right] = 0,
\]
Stability limits of flames propagating downwards

non-dimensional parameters \( \kappa \equiv kd_L \) \( \kappa_m \equiv 1/(2B) \) \( G_0 \equiv (\rho_b/\rho_u)Fr^{-1} \) \( Fr^{-1} \equiv |g|d_L/U_L^2 \)

\[
s = \sigma \tau_L \Rightarrow \left(1 + \frac{1}{v_b^{-1}}\right)s^2 - 2\kappa s - \left(v_b - 1\right)\kappa \left[ -G_o + \kappa \left( 1 - \frac{\kappa}{\kappa_m} \right) \right] = 0
\]

marginal wavenumber \( \sigma = 0 \) \( \left[ -G_o + \kappa \left( 1 - \frac{\kappa}{\kappa_m} \right) \right] = 0, \)

gravity stabilizes the large wavelengths of slow propagating flame

curvature stabilizes the small wavelengths

g gravity stabilizes the large wavelengths of slow propagating flame \( U_L < 10\text{cm/s} \)

instability threshold \( U_L \approx 10\text{cm/s} \)

\[
G_{oc} = \frac{k_c d_L}{2}, \quad k_c = \frac{k_m}{2}, \quad U_{Lc} = \sqrt{\frac{2 \rho_b |g|}{\rho_u k_c}}
\]

OK with experiments by Boyer Quinard and Searby (1982)

Flames propagating upwards: bubble flames