Lectures on
Dynamics of Gaseous Combustion Waves
(from flames to detonations)
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Lecture IX
Turbulent flames
Lecture 9: Turbulent flames

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IX-1) Introduction

The problem of premixed flames in a turbulent flow is still widely open.

Experiments are difficult. Experimental data are very scattered.

The simplest model has no known solution (Nonlinear stochastic equation).

**Reaction-diffusion wave in a turbulent flow** (no gas expansion)

\[
\frac{\partial \theta}{\partial t} + \mathbf{v}(\mathbf{r}, t) \cdot \nabla \theta - D_T \Delta \theta = \omega'(\theta)/\tau_{rb}.
\]

prescribed turbulent flow (stochastic field)

Same model in the wrinkled flame regime \((l_{\text{tur}} \gg d_L, \tau_{\text{tur}} \gg \tau_L \Rightarrow U_n = U_L)\)

**stochastic eikonal eq.**

\[
\frac{\partial G}{\partial t} + \mathbf{v}(\mathbf{r}, t) \cdot \nabla G = U_n |\nabla G|
\]

\[
\vartheta = (u, w_y, w_z)
\]

\[
x = \alpha(y, z, t)
\]

\[
G - G_0 = x - \alpha(y, z, t)
\]

\[
\frac{\partial \alpha}{\partial t} - u(\mathbf{r}_f, t) + w(\mathbf{r}_f, t) \cdot \nabla_{yz} \alpha = U_{\text{tur}} - U_n \sqrt{1 + |\nabla_{yz} \alpha|^2}
\]

\[
\langle S \rangle = \iint dxdy \left\langle \sqrt{1 + |\nabla_{yz} \alpha|^2} \right\rangle \quad U_{\text{tur}}/U_L = \left\langle \sqrt{1 + |\nabla_{yz} \alpha|^2} \right\rangle
\]

The very existence of \(\langle S \rangle\) and \(d_{\text{tur}}\) is questionable.

\[
|\mathbf{v}| \ll U_L \Rightarrow U_{\text{tur}}/U_L \approx 1 + (|\mathbf{v}|/U_L)^2 \quad |\mathbf{v}| \gg U_L \Rightarrow U_{\text{tur}} \approx |\mathbf{v}|
\]

(Shchelkin 1943, Clavin Williams 1979)  

(Bending effect)

\[
\text{modification to the laminar flame structure}
\]

(Damköler 1940)
IX-2) Turbulent diffusion

**Taylor’s diffusion coefficient** (analogy with Einstein random walk for molecular diffusion)

1-D for simplicity: \( \frac{dx}{dt} = v(t), \quad x(t) = \int_{0}^{t} v(t')dt' \)  

\[
\langle x^2(t) \rangle = \int_{0}^{t} dt' \int_{0}^{t} dt'' \langle v(t')v(t'') \rangle 
\]

\[
\langle x^2(t) \rangle = 2 \int_{0}^{t} dt' \int_{0}^{t} d\tau \langle v(t')v(t' - \tau) \rangle 
\]

turbulence: homogeneous in time \( \langle v(t)v(t - \tau) \rangle = \langle v^2 \rangle g(\tau) \quad g(0) = 1, \quad \lim_{\tau \to \infty} g = 0 \)

integration by parts \( \langle x^2(t) \rangle = 2 \langle v^2 \rangle \int_{0}^{t} (t - \tau)g(\tau)d\tau \)  

where \( \int_{0}^{\infty} \tau g(\tau)d\tau = O(\tau_I^2) \quad t \gg \tau_I : \quad g = 0 \)

\[
D_{\text{tur}} \equiv \langle v^2 \rangle \tau_I \quad \langle v^2 \rangle = (\text{turbulence intensity})^2 
\]

**Rough model for the turbulent transport** (analogy with molecular diffusion)

\[
\langle v \theta \rangle \approx -D_{\text{tur}} \nabla \langle \theta \rangle, \quad \langle \nabla . (v \theta) \rangle \approx -D_{\text{tur}} \Delta \langle \theta \rangle 
\]

limited to scalar mixing with small displacement / size (blobs, sheets ..) \( l_I \ll L \quad \langle v_I \rangle \approx \langle v^2 \rangle^{1/2} \quad l_I \equiv v_I \tau_I \)

**Well-stirred flame regime of Damköhler** (1940) \( l_I \ll d_L \quad \text{and} \quad D_{\text{tur}} \gg D_T \)

little practical importance

\[
U_{\text{tur}} \approx \sqrt{D_{\text{tur}}/\tau_b} \quad \frac{U_{\text{tur}}}{U_L} \approx \sqrt{\left( \frac{l_I}{d_L} \right) \left( \frac{v_I}{U_L} \right)} \gg 1, \quad d_{\text{tur}} \approx D_{\text{tur}}/U_{\text{tur}} \gg d_L 
\]
IX-3) Strongly corrugated flamelets regime

Kolmogorov’s cascade (homogeneous, isotropic and fully-developed turbulence)

Decomposition into a sum of vortices of different length and time scales

\[ l_i, \tau_i, v_i \equiv l_i/\tau_i \]

turn-over velocity \hspace{1cm} local Reynolds nb \hspace{1cm} \( \nu \equiv \mu/\rho \)

Kolmogorov scale \( l_K, \tau_K, v_K \quad \text{Re}_K = 1 \quad l_i > l_K \quad v_i > v_K \quad \forall i \)

Integral scale \( l_I, \tau_I, v_I \quad \text{Re}_I \gg 1 \quad l_i > l_I \quad v_I > v_i \quad \forall i \)

Scaling laws (dimensional analysis) \( l_K \ll l_i \ll l_I \)

energy transfer in NS eqs: \( \rho (v \cdot \nabla) v^2/2 \quad v_i^3/l_i \equiv \epsilon \approx \text{cst} \Rightarrow v_i \approx \epsilon^{1/3} l_i^{1/3}, \quad v_i^2 \approx \epsilon^{2/3} l_i^{2/3}, \quad \tau_i \approx \epsilon^{-1/3} l_i^{2/3}, \)

dissipation rate of energy: \( \nu \cdot \Delta v \Rightarrow \epsilon = \nu v_K^2/l_K^2 \quad \epsilon = v_i^3/l_I \)

\[ \text{Re}_K \equiv v_K l_K/\nu = 1 \Rightarrow l_I/l_K \approx \text{Re}_K^{3/4}, \quad v_I/v_K \approx \text{Re}_I^{1/4}, \quad \tau_I/\tau_K \approx \text{Re}_I^{1/2} \quad \text{Re}_I \gg 1 \]

energy spectrum: \( \langle v^2 \rangle/2 = \int_0^\infty dk \, E(k) \quad E(k) \approx \epsilon^{2/3} k^{-5/3} \)

**definition of strongly corrugated flames**

\[ v_K \ll U_L \ll v_I \Rightarrow d_L \ll l_K, \quad \tau_L \ll \tau_K \quad \text{no modification to} \quad \text{the laminar flame structure} \]

**Gibson scale** \( l_G \) (Peters 1986)

definition of the Gibson scale: smallest size of the wrinkles on the flame front

\( \tau_i \approx l_i/U_L \Rightarrow v_i \approx U_L \)

\[ l_G \equiv U_L^3/\epsilon \Rightarrow l_K \ll l_G \ll l_I \]

many scales of wrinkles \( l_G \ll l_I \Rightarrow \text{fractal geometry of the flame front} \)
Elements of fractal geometry

Total surface area in a cube of size $l_i$, $l_G < l_i < l_I$  
$S_i \approx N_{i,G} l_G^2$.  

nb of cubes of size $l_G$ that intersect the surface within the volume $l_i^3$

Weaker resolution $l_j$, $l_G < l_j < l_i$  
$S_{i,j} \approx N_{i,j} l_j^2$.  

nb of cubes of size $l_j$ that intersect the surface within the volume $l_i^3$

Details of small scales are lost as the size of the box $l_j$ increases

$S_{i,j+k} < S_{i,j} < S_{i} \quad N_{i,j+k} < N_{i,j} < N_{i}$

Fractal dimension $D_f > 2 : \quad N_{i,j} \approx (l_i/l_j)^{D_f}$,  

$S_{i,j}/l_i^2 \approx (l_i/l_j)^{D_f-2}$

Regular surface: $D_f = 2 \Rightarrow$ total area $S_i$ in a box of size $l_i$.  
$S_i/l_i^2 = \lim_{l_j \to 0} S_{i,j}/l_i^2 = \text{finite cst.}$

For a flame of thickness $d_L$ its area is well defined for wrinkles whose scale is larger than $d_L$, $l_j > d_L$

The fractal dimension $D_f > 2$ can concern only scales greater than the smallest wrinkles

Fractal dimension of a turbulent flame can be meaningful only for  
$d_L < l_G < l_j < l_i < l_I$
Self similarity of strongly corrugated flames

Assumption: the Kolmogorov cascade is not modified by gas expansion

Contamination time vs combustion time

Kolmogorov cascade \[ \tau_i \approx \epsilon^{-1/3} l_i^{2/3} \backslash l_i \backslash, \quad v_i \approx \epsilon^{1/3} l_i^{1/3} \uparrow l_i \uparrow \]

Fastest contamination: integral scale \( v_I \gg v_i \), \( U_{tur} = v_I \)?

ok if the combustion time of the vortex is not longer than the turnover time

Self similar law

An effective front of thickness \( l_i \) is defined at each scale

A flame velocity \( U_i \) can be defined at each scale if \( U_i = (S_{i,j}/l_i^2) U_j \)

At the Gibson scale the combustion time of the vortex = turnover time \( U_L = v_G \)

Self similarity: same law at all scales \( \Rightarrow \) combustion time of the vortex = the turnover time \( \forall l_i \)

\[ l_i/U_i = \tau_i \Rightarrow U_i = v_i \]

Kolmogorov cascade \( \Rightarrow \) small vortices burn faster than larger ones

\[ U_{tur} = v_I, \quad l_{tur} = l_I \]

Fractal dimension of the flame surface:

\[ U_i/U_j = \langle S_{i,j} \rangle / l_i^2 \Rightarrow u_i/u_j = \langle S_{i,j} \rangle / l_i^2 \Rightarrow \langle S_{i,j} \rangle / l_i^2 = (l_i/l_j)^{1/3} \Rightarrow D_f = 7/3 \]

\[ \langle S_{i,j} \rangle / l_i^2 = (l_i/l_j)^{D_f-2} \]

The result is the same for all mixtures...??
Co-variant laws
Pocheau 1994

More general law independent of the turbulent scaling and satisfying additivity

Turbulent energy contained in the range $[l_i, l_j]$:

$$v_{i,j}^2 = \sum_{k=i}^{j-1} v_k^2$$

$\forall v_k^2$ : energy in $[k, k+1]$.

Co-variant law = same for each couple of length scales $l_i, l_j$ \quad $l_i > l_j$

The only co-variant law for the flame velocity $U_i$ at scale $l_i$ satisfying additivity is

$$U_i^2 = U_j^2 + c v_{i,j}^2$$

Co-variance \quad $l_i > l_k > l_j$, \quad $v_{i,j}^2 = v_{i,k}^2 + v_{k,j}^2$ \quad $U_i^2 = U_j^2 + c v_{i,k}^2 + c v_{k,j}^2 = U_k^2 + c v_{i,k}^2$ Pocheau 1994

$$v^2 \equiv \sum v_n^2$$  turbulence intensity

Not limited to a strong turbulence

The case $c = 1$ covers the known results at low and large turbulence intensity

Reasonably good agreement with experiments

$$v/U_L = O(1), \quad l_I/l_K \approx 180$$
IX-4) Turbulent combustion noise

wavelength $a/\omega \gg L$ size of the flame

Monopolar sound emission

Deformable (small) body with fluctuating volume $V(t)$

\[ u = \nabla \phi(r, t) \quad \text{acoustic potential} \quad \phi(r, t) = -\frac{\dot{V}(t - ar)}{4\pi r} \quad r \equiv |r|, \quad \dot{V}(t) \equiv \frac{dV}{dt} \]

\[ r \gg L : \quad v = (4\pi ar)^{-1}\dot{V}(t - r/a), \quad \ddot{V}(t) \equiv \frac{d^2V(t)}{dt^2} \]

Radiated flux of energy (intensity of sound) \[ I \equiv \rho a \langle v^2 \rangle \]

\[ I = (\rho/4\pi a) \left\langle \left(\frac{d^2V}{dt^2}\right)^2 \right\rangle \]

mass flow rate of burned gas in the lab frame

Sound generated by a turbulent flame \[ \frac{dV}{dt} = \frac{\dot{M}_b}{\rho_b} \]

mass flow rate of fresh gas in the lab frame

\[ \dot{M}_b = \rho_b \int_S (D_f + U_b)d^2\sigma \quad \dot{M}_u = \rho_u \int_S (D_f + U_L)d^2\sigma \quad \rho_u U_L = \rho_b U_b \]

normal flame velocity in the lab frame

elimination of $D_f$ \[ \frac{dV}{dt} = \frac{\dot{M}_b}{\rho_b} = \frac{\dot{M}_u}{\rho_u} + (U_b - U_L)S \]

constant

\[ I = (\rho/4\pi a)(U_b - U_L)^2 \left\langle (dS/dt)^2 \right\rangle \]

intensity of sound

Strahle 1985

\[ \frac{dI(\omega)}{d\omega} = \frac{\rho}{4\pi a}(U_b - U_L)^2 \int_0^\infty dt e^{i\omega t} \left\langle \dot{S}(t)\dot{S}(0) \right\rangle \]

power spectrum of sound

\[ \dot{M}_b = \rho_b \int_S (D_f + U_b)d^2\sigma \quad \dot{M}_u = \rho_u \int_S (D_f + U_L)d^2\sigma \quad \rho_u U_L = \rho_b U_b \]

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Strahle 1985

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power spectrum of sound
$I = (\rho/4\pi a)(U_b - U_L)^2 \langle (dS/dt)^2 \rangle$

The intensity of sound

$\frac{d\tilde{I}(\omega)}{d\omega} = \frac{\rho}{4\pi a} (U_b - U_L)^2 \int_0^\infty dt e^{i\omega t} \langle \dot{S}(t) \dot{S}(0) \rangle$

The power spectrum of sound

Stongly corrugated regime with a Kolmogorov cascade:

$d\tilde{I}(\omega) \propto \omega^{-5/2} d\omega$

in agreement with experiments on very large burners

in agreement with experiments on very large burners

Clavin Siggia 1991

Blowtorch noise

Combustion noise is two orders of magnitude higher

the noise is not resulting from the direct interaction of upstream turbulence on the flame front

amplification by the intrinsic flame instability is essential

(Searby 2001)