Laminar Premixed Flames: Kinematics and Burning Velocity

CEFRC Combustion Summer School

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Course Overview

Part I: Fundamentals and Laminar Flames

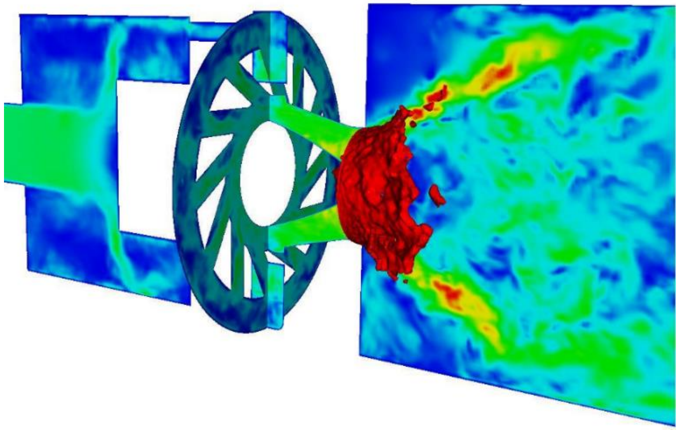
- Introduction
- Fundamentals and mass balances of combustion systems
- Thermodynamics, flame temperature, and equilibrium
- Governing equations
- Laminar premixed flames: Kinematics and Burning Velocity
- Laminar premixed flames: Flame structure
- Laminar diffusion flames

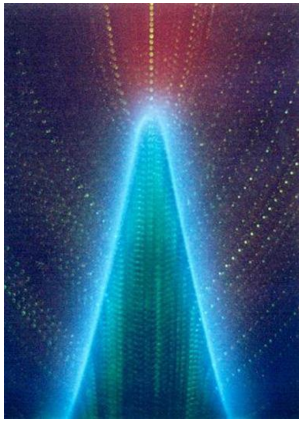
- Introduction
- Kinematic balance for steady oblique flames
- Laminar burning velocity
- Field equation for the flame position
- Flame stretch and curvature
- Thermal-diffusive flame instability
- Hydrodynamic flame instability

Laminar Premixed Flames

- Premixed combustion used in combustion devices when high heat release rates are desired
  - Small devices
  - Low residence times
- Examples:
  - SI engine
  - Stationary gas turbines
- Advantage → Lean combustion possible
  - Smoke-free combustion
  - Low NO_x
- Disadvantage: Danger of
  - Explosions
  - Combustion instabilities

    → Large-scale industrial furnaces and aircraft engines are typically non-premixed

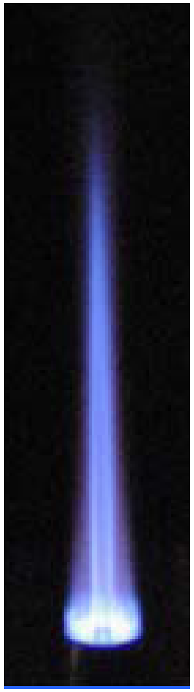




Premixed Flames

- Premixed flame: Blue or blue-green by chemiluminescence of excited radicals, such as C_2^o and CH^o
- Diffusion flames: Yellow due to soot radiation

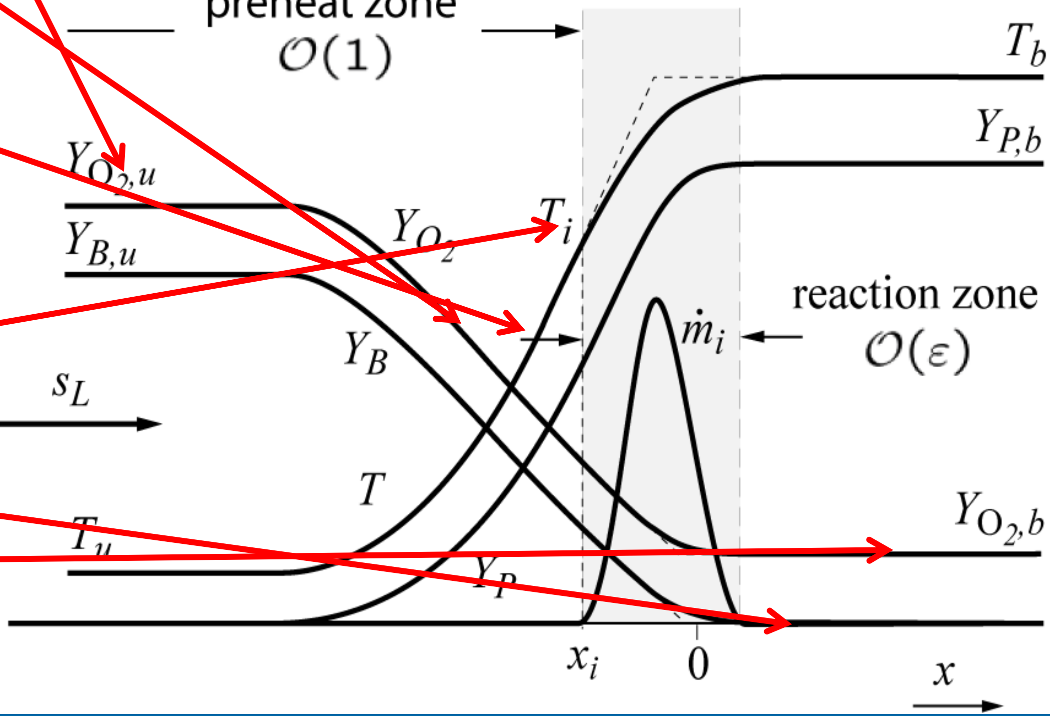
Laminar Bunsen Flame



Turbulent Premixed Flame (Dunn et al.)

Flame Structure of Premixed Laminar Flames

- Fuel and oxidizer are convected from upstream with the burning velocity s_L
- Fuel and air diffuse into the reaction zone
- Mixture heated up by heat conduction from the burnt gases
- Fuel consumption, radical production, and oxidation when inner layer temperature is reached
- Increase temperature and gradients
- Fuel is entirely depleted
- Remaining oxygen is convected downstream



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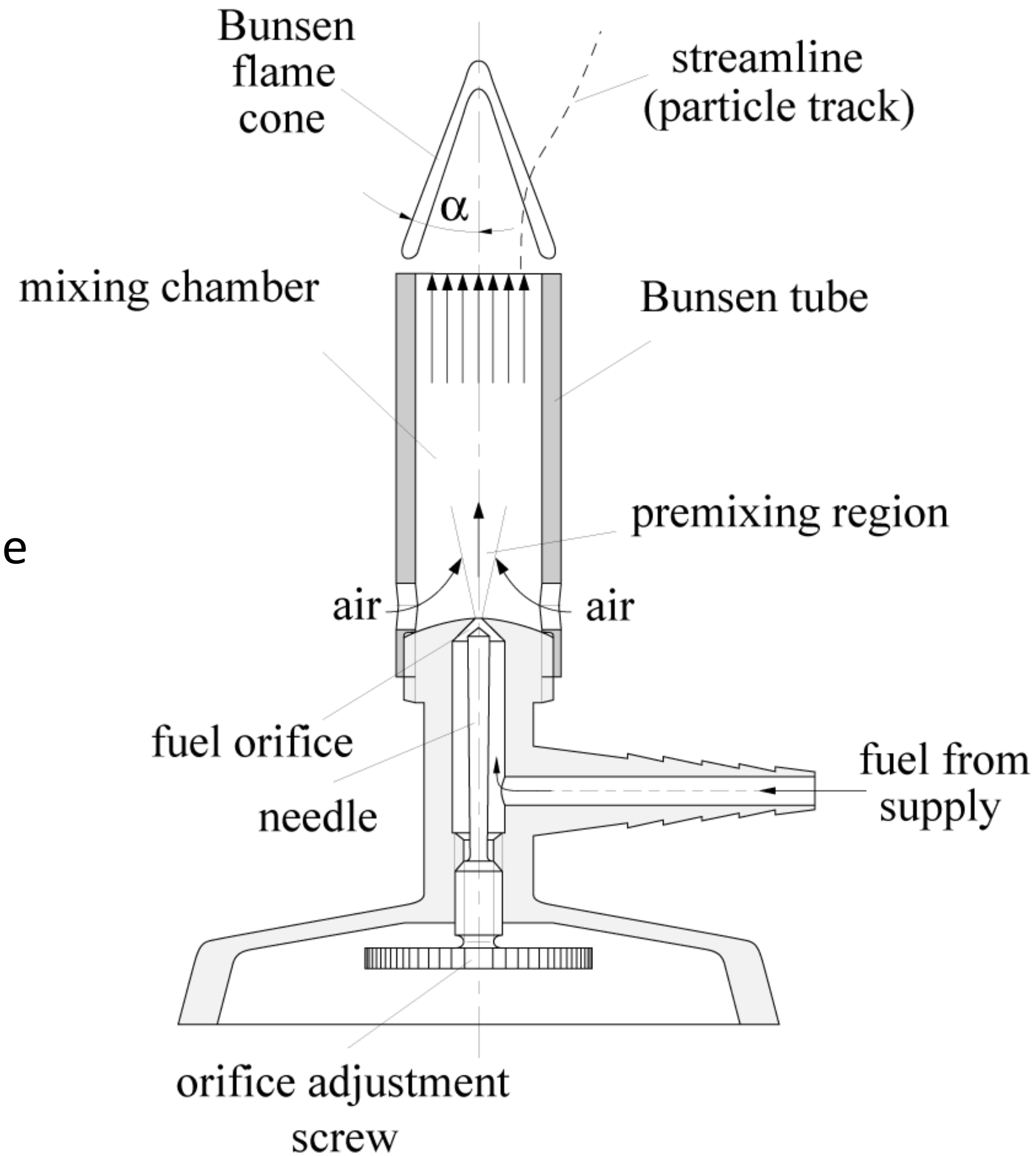
Premixed Flame in a Bunsen Burner

- Fuel enters the Bunsen tube with high momentum through a small orifice
- High momentum
  → underpressure
  → air entrainment into Bunsen tube
- Premixing of fuel and air in the Bunsen tube
- At tube exit: homogeneous, premixed fuel/air mixture, which can and should(!) be ignited

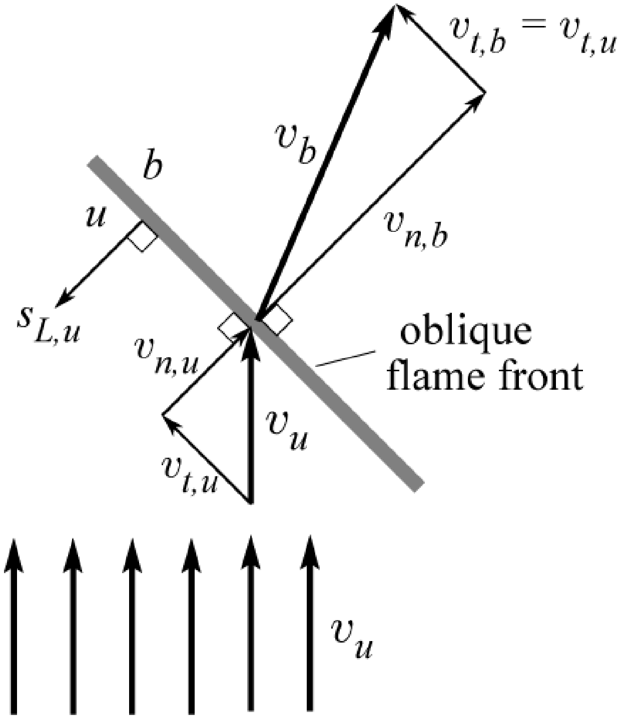


Kinematic Balance for Steady Oblique Flame

- In steady state, flame forms Bunsen cone

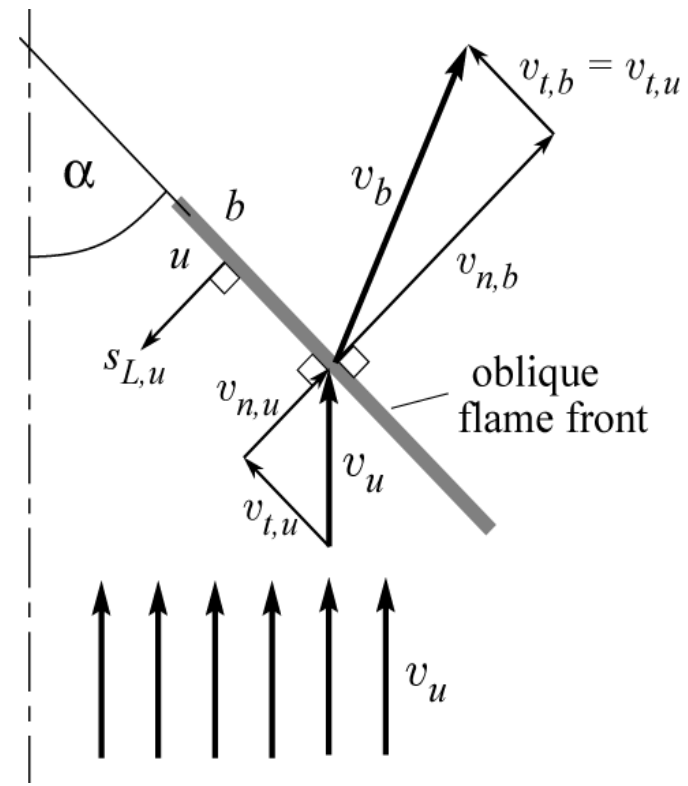
- Velocity component normal to flame front is locally equal to the propagation velocity of the flame front

  → Burning velocity



Kinematic Balance for Steady Oblique Flame

- Laminar burning velocity $s_{L,u}$: Velocity of the flame normal to the flame front and relative to the unburnt mixture (index 'u')
- Can principally be experimentally determined with the Bunsen burner
- Need to measure
  - Velocity of mixture at Bunsen tube exit
  - Bunsen cone angle α

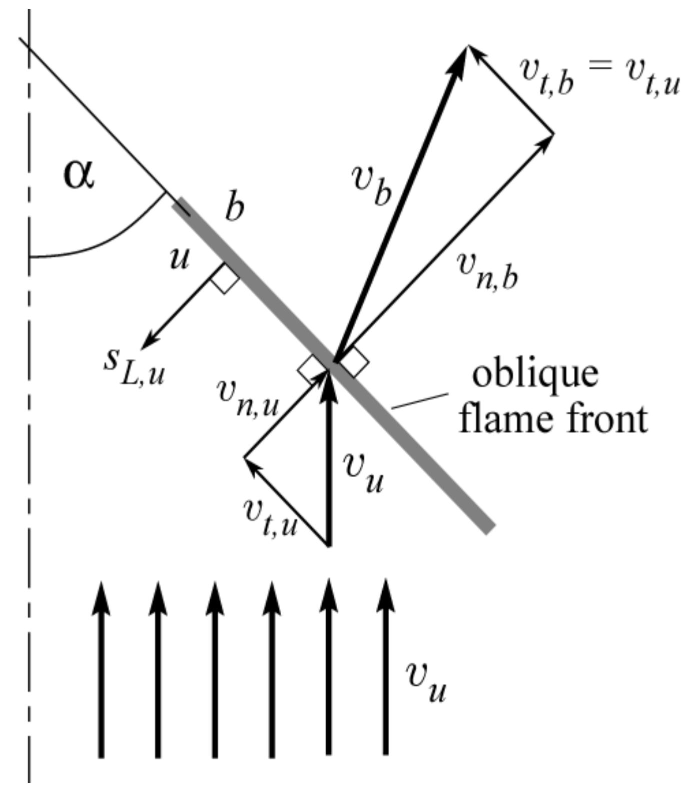


Kinematic Balance for Steady Oblique Flame

- Splitting of the tube exit velocity in components normal and tangential to the flame
- Kinematic balance yields relation unburnt gas velocity and flame propagation velocity

- For laminar flows:

$$s_{L,u} = v_{n,u} = v_u \sin\alpha$$



Kinematic Balance for Steady Oblique Flame

- Flame front:
  - Large temperature increase
  - Pressure almost constant

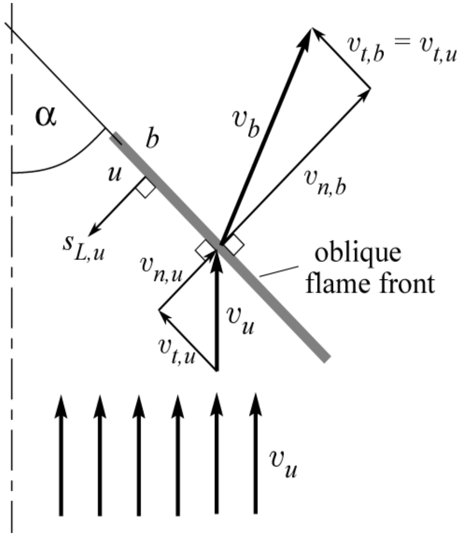
    → Density decreases drastically
- Mass balance normal to the flame front:

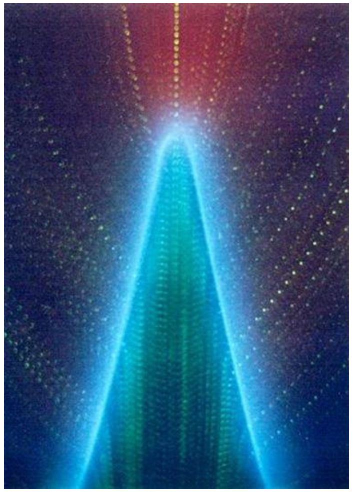
$$(\rho v_n)_u = (\rho v_n)_b \quad \Rightarrow \quad v_{n,b} = v_{n,u}\frac{\rho_u}{\rho_b}$$

- Normal velocity component increases through flame front
- Momentum balance in tangential direction:

$$v_{t,u} = v_{t,b}$$

→ Deflection of the streamlines away from the flame



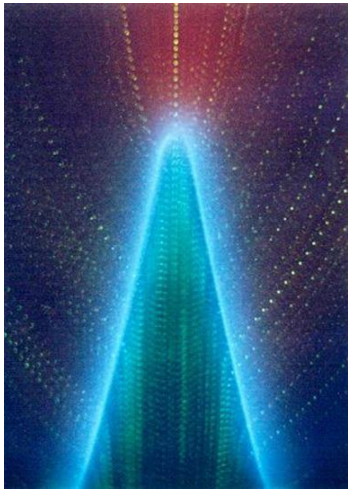


Laminar Bunsen flame (Mungal et al.)

Burning Velocity at the Flame Tip

- Tip of the Bunsen cone
  - Symmetry line
  - Burning velocity equal to velocity in unburnt mixture
  - Here: Burning velocity = normal component, tangential component = 0

  → Burning velocity at the tip by a factor $1/\sin(\alpha)$ larger than burning velocity through oblique part of the cone



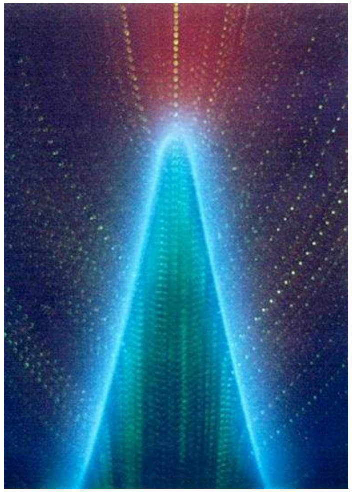
Laminar Bunsen flame (Mungal et al.)

Burning velocity at the flame tip

- Explanation: Strong curvature of the flame front at the tip

  → Increased preheating

  - In addition to heat conduction normal to the flame front preheating by the lateral parts of the flame front



Laminar Bunsen flame
(Mungal et al.)

- Effect of non-unity Lewis numbers

  → Explanation of difference between lean hydrogen and lean hydrocarbon flames

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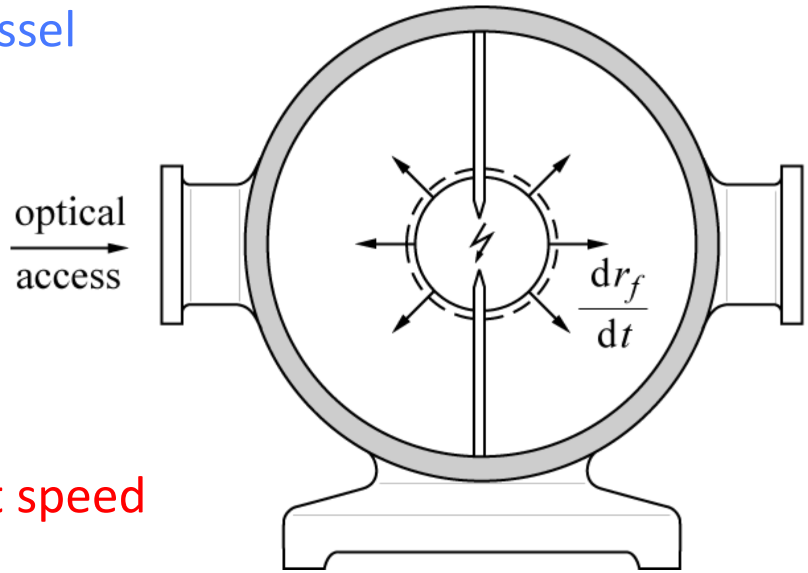
Measuring the laminar burning velocity

- Spherical constant volume combustion vessel
  - Flame initiated by a central spark
  - Spherical propagation of a flame
  - Measurements of radial flame propagation velocity dr_f / dt

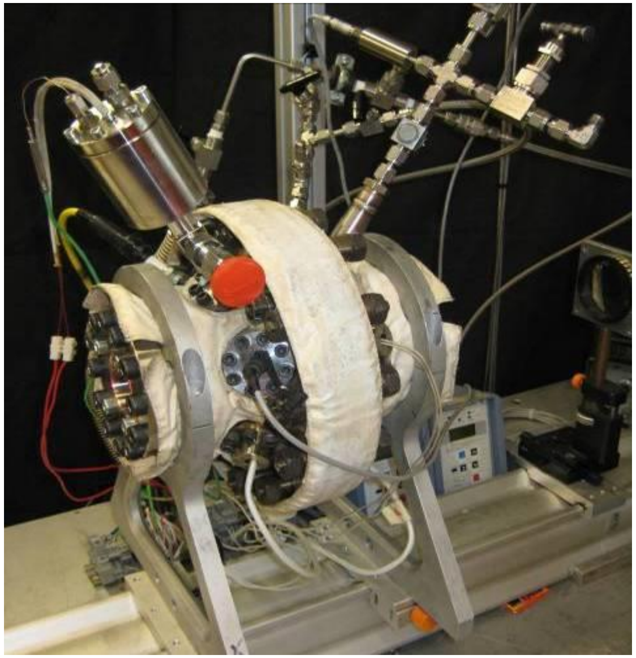


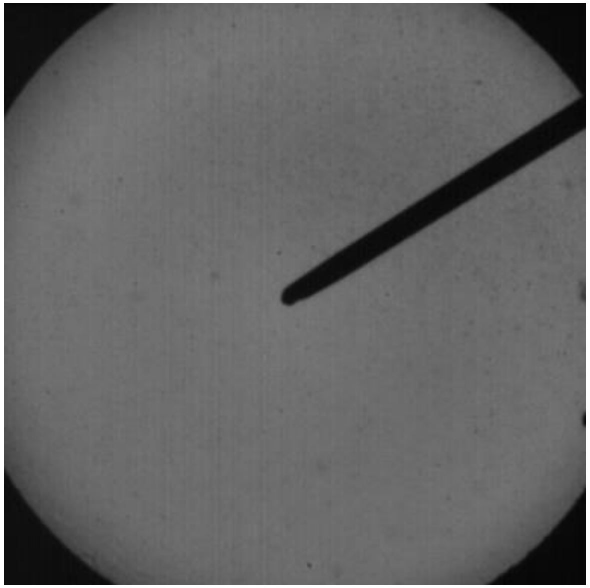
- Kinematic relation for flame displacement speed

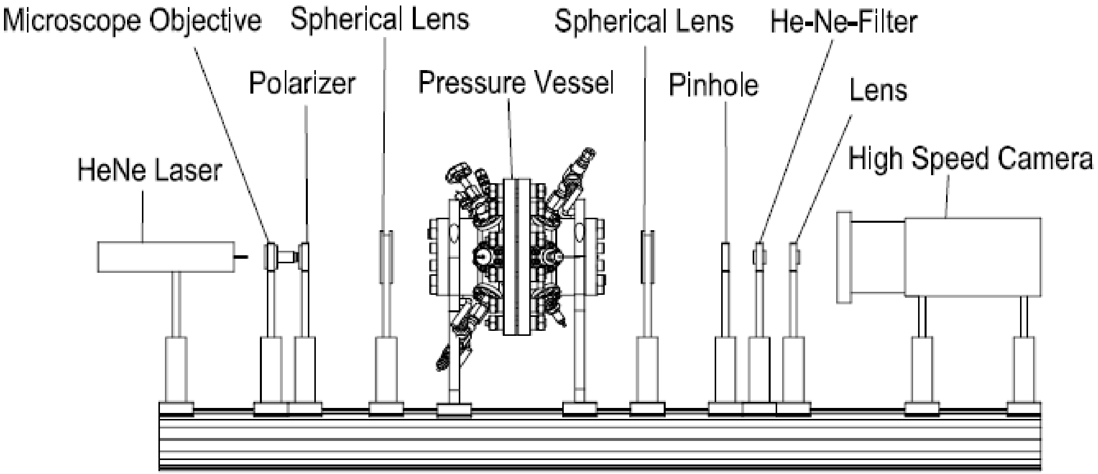
$$\frac{dr_f}{dt} = v_u + s_{L,u}$$

- Flame front position and displacement speed are unsteady
- Pressure increase negligible as long as volume of burnt mixture small relative to total volume
- Influence of curvature

Measuring the laminar burning velocity

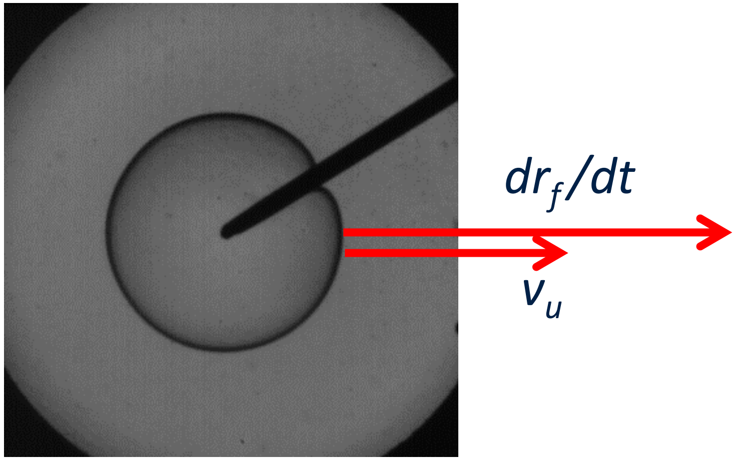






Flame front velocity in a spherical combustion vessel

- Velocity relative to flame front is the burning velocity
  - Different in burnt and unburnt region
- From kinematic relation



$$\frac{dr_f}{dt} = v_u + s_{L,u}$$

  - Velocity on the unburnt side (relative to the flame front) $v_u - dr_f/dt$
  - Burnt side of the front $v_b - dr_f/dt$

- Spherical propagation: Due to symmetry, flow velocity in the burnt gas is zero

$$v_b = 0$$

- Mass balance yields:

$$\rho_u(v_u - \frac{dr_f}{dt}) = \rho_b(v_b - \frac{dr_f}{dt}) = \rho_b(-\frac{dr_f}{dt})$$

Flame front velocity in a spherical combustion vessel

- From mass balance and kinematic relation follows

$$\frac{dr_f}{dt} = \frac{\rho_u}{\rho_u - \rho_b} v_u = v_u + s_{L,u}$$

- Flow velocity on the unburnt side of the front

$$v_u = \frac{\rho_u - \rho_b}{\rho_b} s_{L,u}$$

→ Flow of the unburnt mixture induced by the expansion of the gases behind the flame front

- Measurements of the flame front velocity dr_f/dt

  → Burning velocity $s_{L,u}$:

$$s_{L,u} = \frac{\rho_b}{\rho_u} \frac{dr_f}{dt}.$$

Relation between $s_{L,u}$ and $s_{L,b}$

- Burning velocity $s_{L,u}$ defined with respect to the unburnt mixture
- Another burning velocity $s_{L,b}$ can be defined with respect to the burnt mixture
- Continuity yields the relation:

$$s_{L,b} = \frac{\rho_u}{\rho_b} s_{L,u}$$

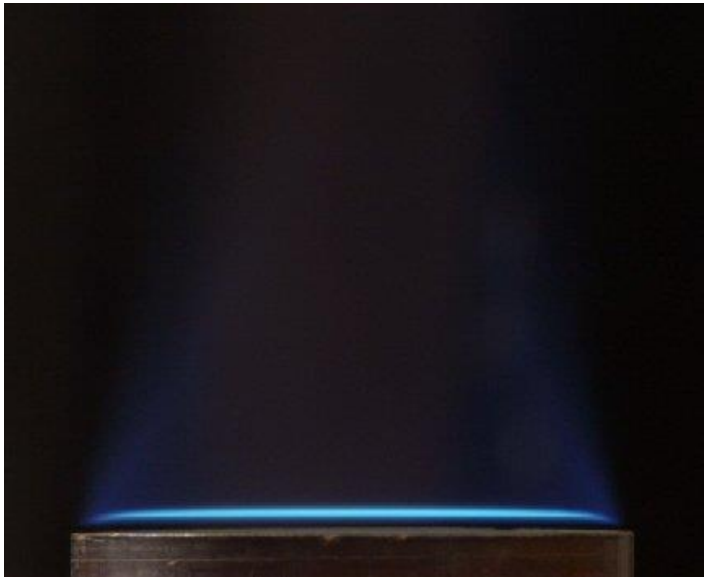
- In the following, we will usually consider the burning velocity with respect to the unburnt $s_L = s_{L,u}$

Flat Flame Burner and Flame Structure

- One-dimensional flame
- Stabilization by heat losses to burner

- In theory, velocity could be increased until heat losses vanish, then
  - → unstretched
  - → $u_u = s_L$

- Analysis of flame structure of flat flames
  - Measurements of temperature and species concentration profiles



The general case with multi-step chemical kinetics

- Laminar burning velocity s_L can be calculated by solving governing conservation equations for the overall mass, species, and temperature (low Mach limit)

- Continuity

$$\frac{\mathrm{d}(\rho u)}{\mathrm{d}x} = 0$$

- Species

$$\rho u \frac{\mathrm{d}Y_i}{\mathrm{d}x} = -\frac{\mathrm{d}j_i}{\mathrm{d}x} + \dot{m}_i$$

- Energy

$$\rho u c_p \frac{\mathrm{d}T}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(\lambda \frac{\mathrm{d}T}{\mathrm{d}x}) - \sum_{i=1}^{k} h_i \dot{m}_i - \sum_{i=1}^{k} c_p j_i \frac{\mathrm{d}T}{\mathrm{d}x} + \frac{\partial p}{\partial t}$$

The general case with multi-step chemical kinetics

- Continuity equation may be integrated once to yield

$$\rho u = \rho_u s_L$$

- Burning velocity is eigenvalue, which must be determined as part of the solution

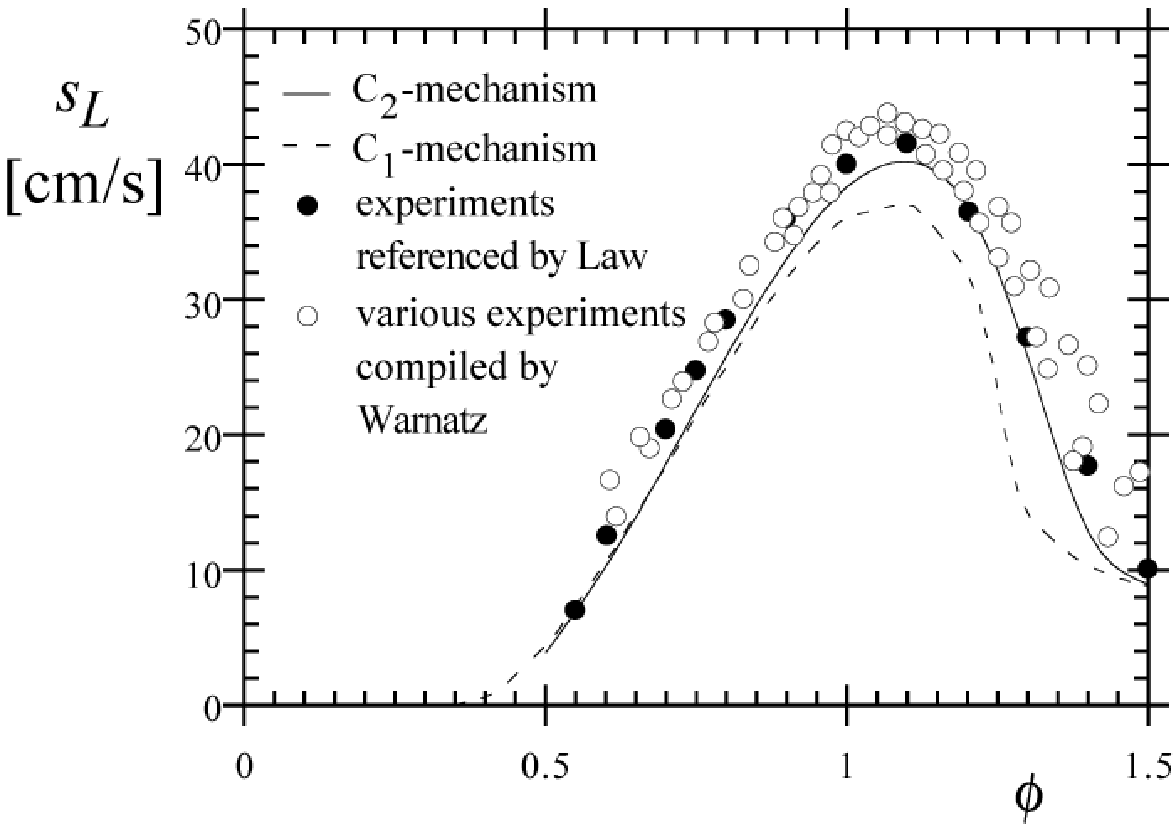
- System of equations may be solved numerically with
  - Appropriate upstream boundary conditions
  - Zero gradient boundary conditions downstream

The general case with multi-step chemical kinetics

- Example: Calculations of the burning velocity of premixed methane-air flames

- Mechanism that contains only C_1-hydrocarbons
  → s_L underpredicted

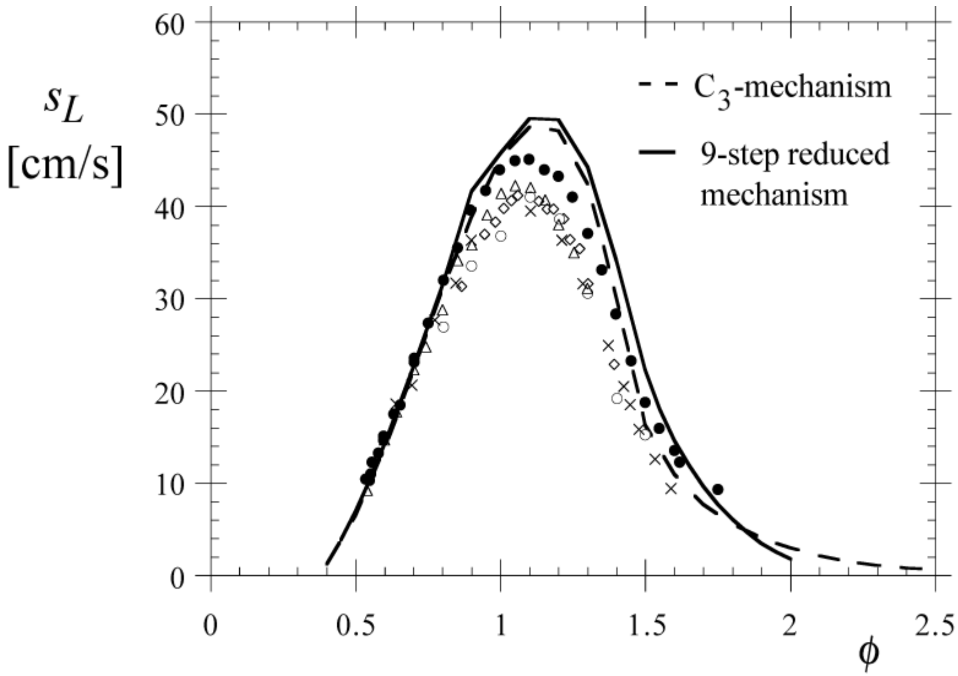
- Including C_2-mechanism [Mauss 1993]
  → Better agreement



The general case with multi-step chemical kinetics

- Example:

  Burning velocities of propane flames taken from Kennel (1993)



- s_L typically decreases with increasing pressure but increases with increasing preheat temperature

Burning Velocity

- Burning velocity is fundamental property of a premixed flame

- Can be used to determine flame dynamics

- Depends on thermo-chemical parameters of the premixed gas ahead of flame only

But:

→ For Bunsen flame, the condition of a constant burning velocity is violated at the tip of the flame

→ Curvature must be taken into account

Next

- We will first calculate flame shapes
- Then we will consider external influences that locally change the burning velocity and discuss the response of the flame to these disturbances

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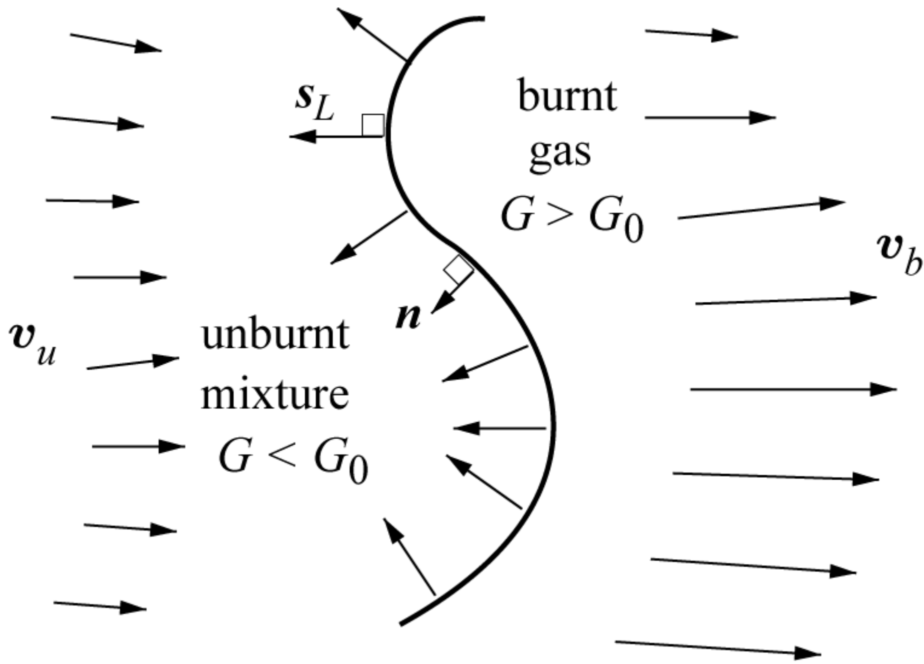
A Field Equation Describing the Flame Position

- Kinematic relation $\frac{\mathrm{d}r_f}{\mathrm{d}t} = v_u + s_{L,u}$ between
  - Displacement velocity $\frac{\mathrm{d}r_f}{\mathrm{d}t}$
  - Flow velocity v_u
  - Burning velocity $s_{L,u}$
- May be generalized by introducing vector ***n*** normal to the flame

$$\frac{\mathrm{d}\boldsymbol{x}_f}{\mathrm{d}t} = \boldsymbol{v} + s_L \boldsymbol{n},$$

where $\boldsymbol{x}_f$ is the vector describing the flame position,

$\mathrm{d}\boldsymbol{x}_f/\mathrm{d}t$ the flame propagation velocity, and $\boldsymbol{v}$ the velocity vector



A Field Equation Describing the Flame Position

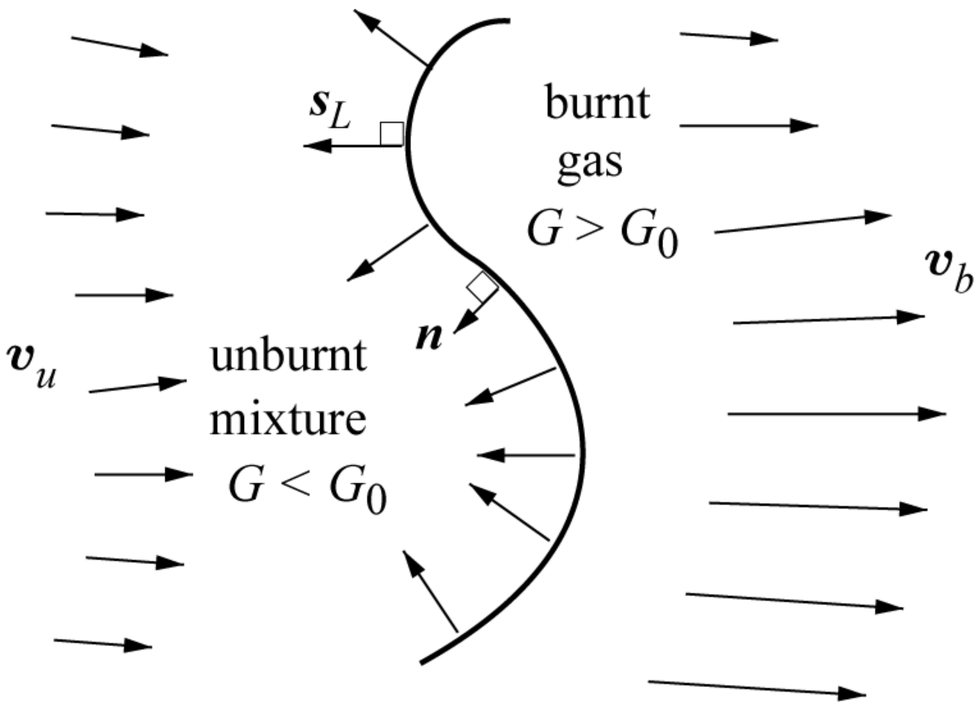
- Normal vector points towards the unburnt mixture and is given by

$$\boldsymbol{n} = -\frac{\nabla G}{|\nabla G|},$$

where $G(\boldsymbol{x},t)$ can be identified as a scalar field whose level surface

$$G(\boldsymbol{x},t) = G_0,$$

represents the flame surface and G_0 is arbitrary



- The flame contour $G(\boldsymbol{x},t) = G_0$ divides physical field into two regions, where $G > G_0$ is the region of burnt gas and $G < G_0$ that of the unburnt mixture

A Field Equation Describing the Flame Position

- Differentiating $G(\boldsymbol{x},t) = G_0$ with respect to t at $G = G_0$ gives

$$\frac{\partial G}{\partial t} + \boldsymbol{\nabla} G \cdot \frac{\partial \boldsymbol{x}}{\partial t}\Big|_{G=G_0} = 0$$

- Introducing $\frac{\mathrm{d}\boldsymbol{x}_f}{\mathrm{d}t} = \boldsymbol{v} + s_L \boldsymbol{n},$ leads to

$$\frac{\partial G}{\partial t} + \boldsymbol{v} \cdot \nabla G = -s_L \boldsymbol{n} \cdot \nabla G$$

- Level set equation for the propagating flame follows using $\boldsymbol{n} = -\frac{\boldsymbol{\nabla} G}{|\boldsymbol{\nabla} G|}$ as

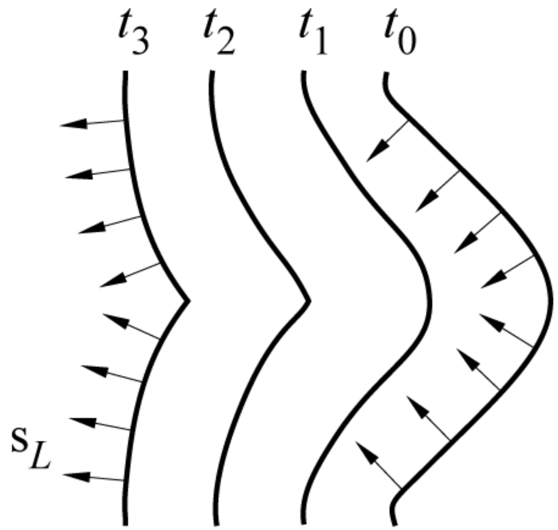
$$\frac{\partial G}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} G = s_L |\boldsymbol{\nabla} G|$$

A Field Equation Describing the Flame Position

- Burning velocity s_L is defined w.r.t. the unburnt mixture
  → Flow velocity $\boldsymbol{v}$ is defined as the conditioned velocity field in the unburnt mixture ahead of the flame
- For a constant value of s_L, the solution of

$$\frac{\partial G}{\partial t} + \boldsymbol{v} \cdot \nabla G = s_L |\nabla G|$$

  is non-unique, and cusps will form where different parts of the flame intersect



- Even an originally smooth undulated front in a quiescent flow will form cusps and eventually become flatter with time
- This is called Huygens' principle

*Exercise: Slot Burner

- A closed form solution of the G-equation

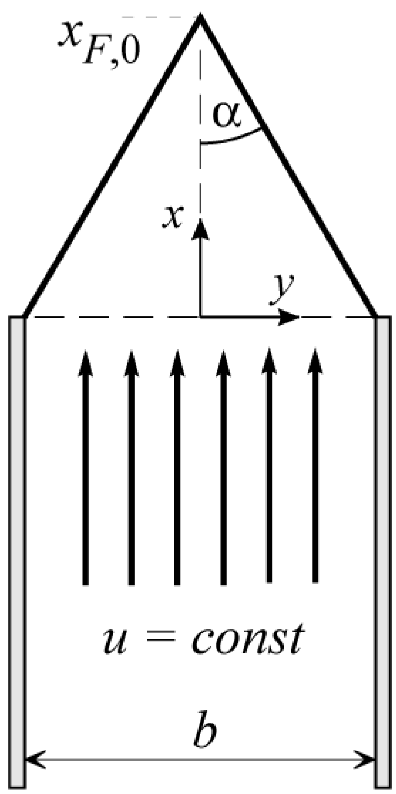
$$\frac{\partial G}{\partial t} + \boldsymbol{v} \cdot \nabla G = s_L |\nabla G|$$

can be obtained for the case of a slot burner with a constant exit velocity *u* for premixed combustion,

- This is the two-dimensional planar version of the axisymmetric Bunsen burner.

- The *G*-equation takes the form

$$u \frac{\partial G}{\partial x} = s_L \sqrt{\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2}$$



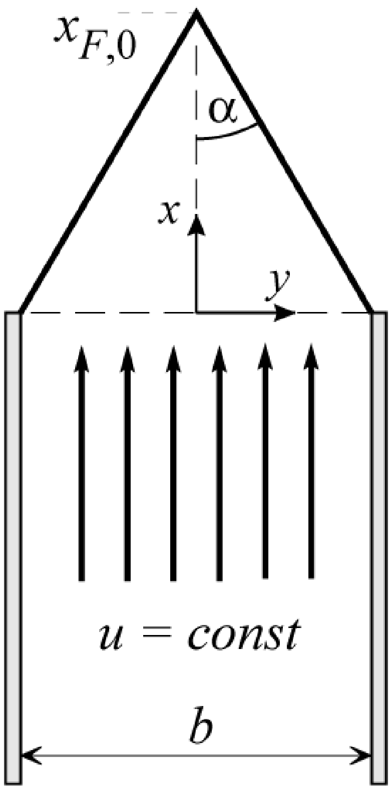
*Exercise: Slot Burner

- With the ansatz $G = x + F(y)$

  and G_0 = 0 one obtains $u = s_L \sqrt{1 + \left(\frac{\partial F}{\partial y}\right)^2}$

  leading to

$$F = \sqrt{\frac{u^2 - s_L^2}{s_L^2}}\,|y| + const.$$



- As the flame is attached at $x = 0$, $y = \pm b/2$, where $G = 0$, this leads to the solution

$$G = \sqrt{\frac{u^2 - s_L^2}{s_L^2}}\left(|y| - \frac{b}{2}\right) + x.$$

*Exercise: Slot Burner

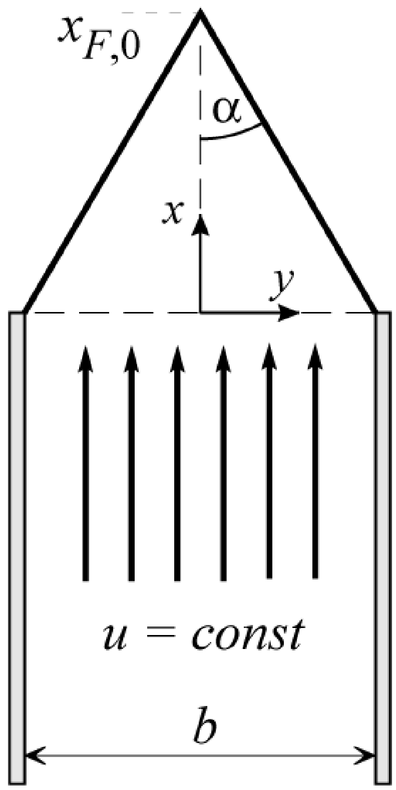
The flame tip lies with *y*=0, *G* = 0 at

$$x_{F,0} = \frac{b}{2}\sqrt{\frac{u^2 - s_L^2}{s_L^2}}$$

and the flame angle a is given by

$$\tan\alpha = \frac{b}{2x_{F,0}} = \sqrt{\frac{u^2 - s_L^2}{s_L^2}}.$$

With $\tan^2\alpha = \sin^2\alpha/(1-\sin^2\alpha)$ it follows that $\sin\alpha = \frac{s_L}{u}$,
which is equivalent to $s_{L,u} = v_{n,u} = v_u \sin\alpha$



This solution shows a cusp at the flame tip $x = x_{F0}$, $y = 0$. In order to obtain a rounded flame tip, one has to take modifications of the burning velocity due to flame curvature into account. This leads to the concept of flame stretch.

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Flame stretch

- Flame stretch consists of two contributions:
  - Flame curvature
  - Flow divergence or strain

- For one-step large activation energy reaction and with the assumption of constant properties, the burning velocity s_L is modified by these two effects as

$$s_L = s_L^0 - s_L^0 \mathcal{L} \kappa + \mathcal{L} \boldsymbol{n} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{n}.$$

- $s^0{}_L$ is the burning velocity for an unstretched flame
- $\mathcal{L}$ is the Markstein length

Flame stretch

- The flame curvature κ is defined as

$$\kappa = \nabla \cdot \boldsymbol{n} = -\nabla \cdot \left(\frac{\nabla G}{|\nabla G|} \right)$$

which may be transformed as

$$\kappa = -\frac{\nabla^2 G + \boldsymbol{n} \cdot \nabla (\boldsymbol{n} \cdot \nabla G)}{|\nabla G|}.$$

- The Markstein length $\mathcal{L}$ appearing in $s_L = s_L^0 - s_L^0 \mathcal{L} \kappa + \mathcal{L} \boldsymbol{n} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{n}.$ is of same order of magnitude and proportional to laminar flame thickness ℓ_F

- Ratio $\ell_F / \mathcal{L}$ is called Markstein number

Markstein length

- With assumptions:
  - One-step reaction with a large activation energy
  - Constant transport properties and heat capacity c_p

  → Markstein length with respect to the unburnt mixture

Unstretched laminar burning velocity

$$s_L = s_L^0 - s_L^0 \mathcal{L}\kappa - \mathcal{L}S$$

- Markstein length
  - Determined experimentally
  - Determined by asymptotic analysis

$$\frac{\mathcal{L}_u}{l_F} = \frac{1}{\gamma}\ln\left(\frac{1}{1-\gamma}\right) + \frac{Ze\,(Le-1)}{2}\frac{(1-\gamma)}{\gamma}\int_0^{\gamma/(1-\gamma)}\frac{\ln(1+x_i)}{x_i}\mathrm{d}x_i$$

Density ratio

Zeldovich-Number

$$Ze = \frac{E}{RT_b}\frac{T_b - T_u}{T_b}$$

Lewis-Number

$$Le = \frac{\lambda}{\rho c_p D} = \frac{Sc}{Pr}$$

Markstein length

- Markstein length

$$\frac{\mathcal{L}_u}{\ell_F} = \frac{1}{\gamma}\ln\frac{1}{1-\gamma} + \frac{\mathrm{Ze}(\mathrm{Le}-1)}{2}\frac{(1-\gamma)}{\gamma}\int_0^{\gamma/(1-\gamma)}\frac{\ln(1+x)}{x}dx\ .$$

- Derived by Clavin and Williams (1982) and Matalon and Matkowsky (1982)
- $\mathrm{Ze} = E(T_b - T_u)/(\mathcal{R}T_b^2)$ is the Zeldovich number, where *E* is the activation energy, $\mathcal{R}$ the universal gas constant, and Le the Lewis number of the deficient reactant
- Different expression can be derived, if both s_L and $\mathcal{L}$ are defined with respect to the burnt gas [cf. Clavin, 1985]

*Example: Effect of Flame Curvature

- We want to explore the influence of curvature on the burning velocity for the case of a spherical propagating flame

- Flow velocity is zero in the burnt gas
  → Formulate the *G*-equation with respect to the burnt gas:
$$\frac{\mathrm{d}r_f}{\mathrm{d}t} = s_{L,b}$$
  where $r_f(t)$ is the radial flame position
- The burning velocity is then $s^0_{L,b}$ and the Markstein length is that with respect to the burnt gas $\mathcal{L}_b$.

- Here, we assume $\mathcal{L}_b > 0$ to avoid complications associated with thermo-diffusive instabilities

*Example: Effect of Flame Curvature

- In a spherical coordinate system, the *G*-equation reads

$$\frac{\partial G}{\partial t} = s_{L,b}^{0} \left(\left| \frac{\partial G}{\partial r} \right| + \frac{2\mathcal{L}_b}{r} \frac{\partial G}{\partial r} \right),$$

where the entire term in round brackets represents the curvature in spherical coordinates

- We introduce the ansatz $G = r_f(t) - r,$

to obtain at the flame front $r=r_f$

$$\frac{\partial r_f}{\partial t} = s_{L,b}^{0} \left(1 - \frac{2\mathcal{L}_b}{r_f} \right).$$

- This equation may also be found in Clavin (1985)

*Example: Effect of Flame Curvature

- This equation reduces to $\frac{\mathrm{d}r_f}{\mathrm{d}t} = s_{L,b}$ for $\mathcal{L}_b = 0$.

- It may be integrated to obtain

$$s_{L,b}^0 t = r_f - r_{f,0} + 2\mathcal{L}_b \ln\left(\frac{r_f - 2\mathcal{L}_b}{r_{f,0} - 2\mathcal{L}_b}\right),$$

  where the initial radius at t=0 is denoted by $r_{f,0}$

- This expression has no meaningful solutions for $r_{f,0} < 2\mathcal{L}_b$, indicating that there needs to be a minimum initial flame kernel for flame propagation to take off

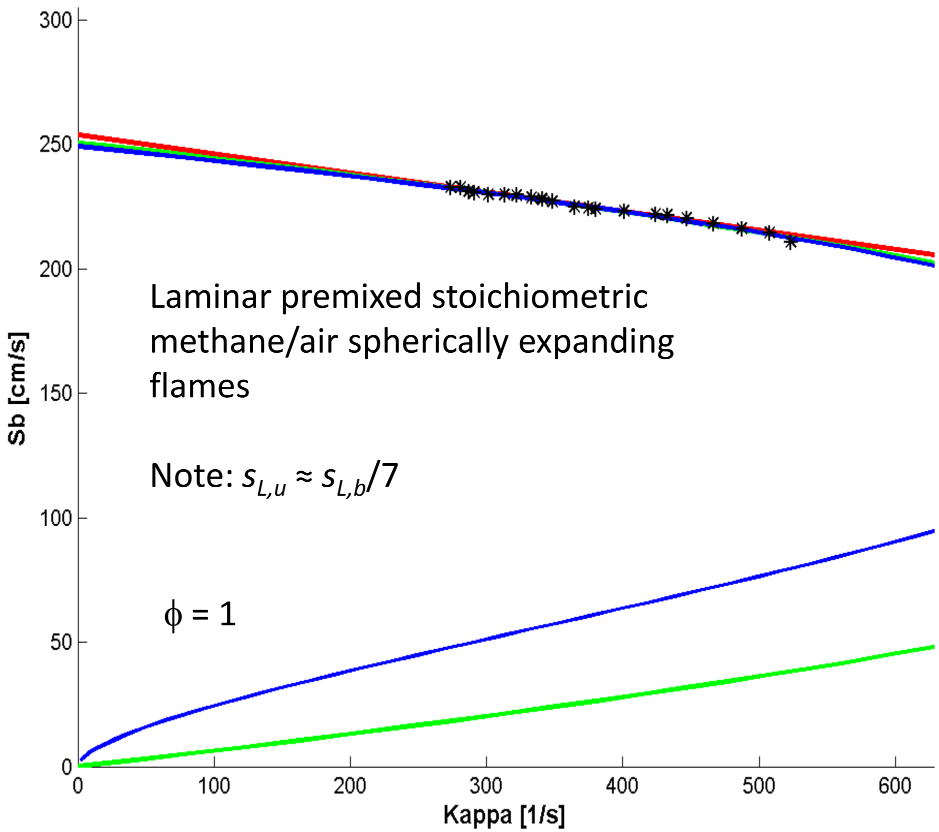
- It should be recalled that $s_L = s_L^0 - s_L^0 \mathcal{L}\kappa + \mathcal{L}\boldsymbol{n}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{n}.$
  is only valid if the product $\mathcal{L}\kappa \ll 1$.

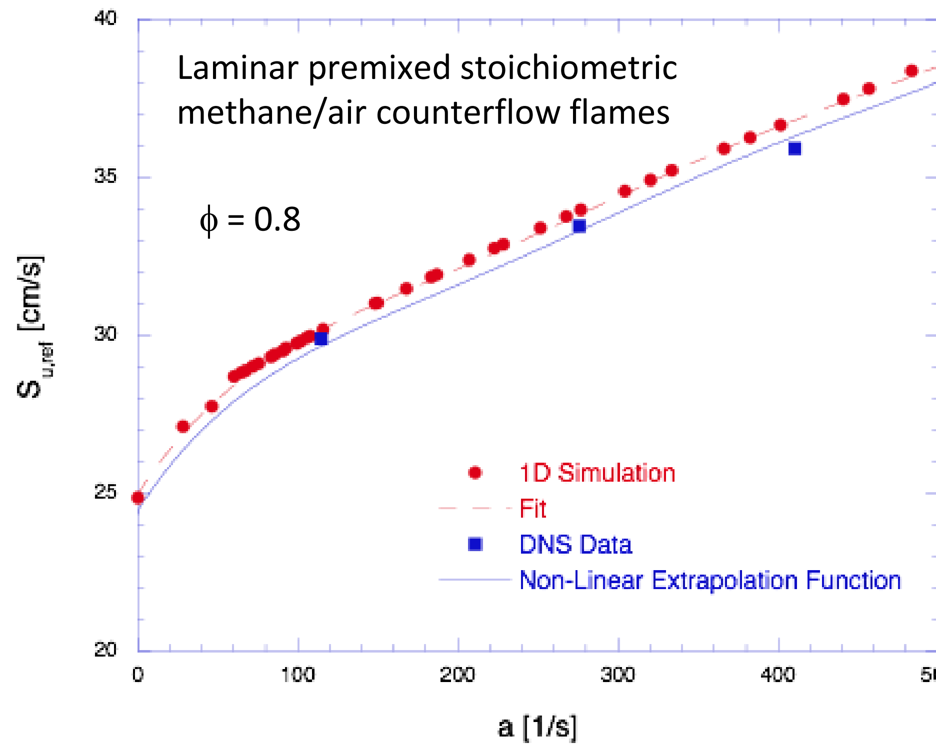
- For $r_{f,0} > 2\mathcal{L}_b$ curvature corrections are important at early times only

Effects of curvature and strain on laminar burning velocity

Curvature Effect on Laminar Burning Velocity from Experiments and Theory

Strain Effect on Laminar Burning Velocity from Numerical Simulations





$$s_L = s_L^0 - s_L^0 \mathcal{L}\kappa - \mathcal{L}S$$

Course Overview

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- Hydrodynamic flame instability

Flame Instabilities: Thermal-diffusive instability

Effect of Curvature

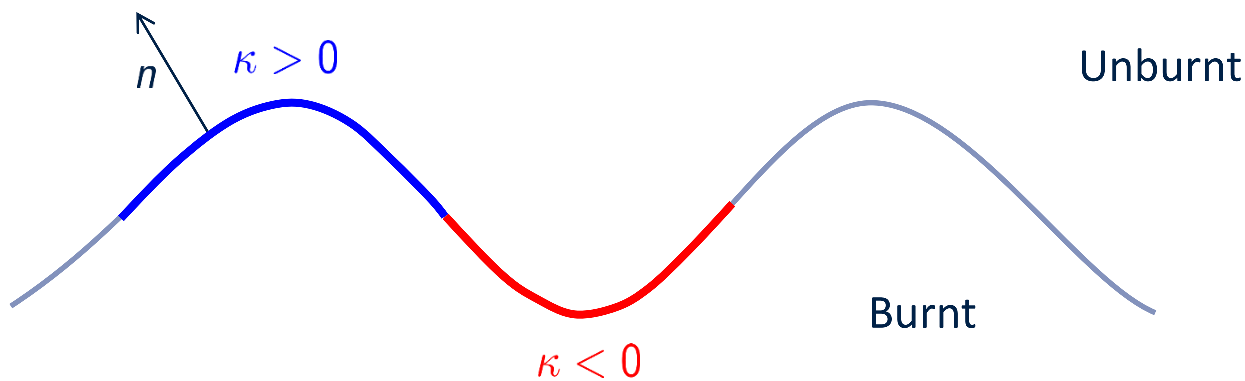
$$\kappa = \frac{\partial n_i}{\partial x_i}$$

Unstretched laminar burning velocity

Effect of stretch

$$S = -n_i \frac{\partial u_i}{\partial x_j} n_j$$

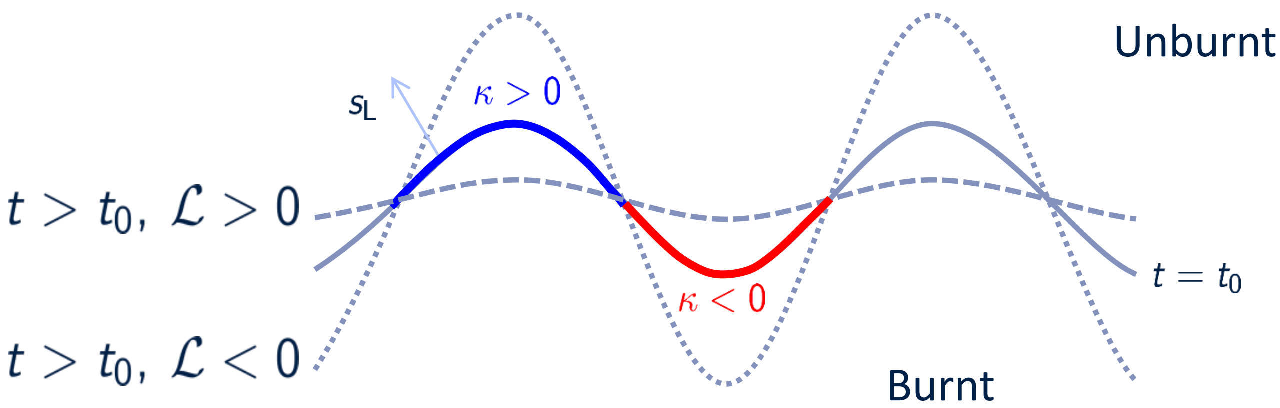
$$s_L = s_L^0 - s_L^0 \mathcal{L}\kappa - \mathcal{L}S$$



Flame Instabilities: Thermal-diffusive instability

Unstretched laminar burning velocity

$$s_L = s_L^0 - s_L^0 \mathcal{L}\kappa - \mathcal{L}\not{S}$$



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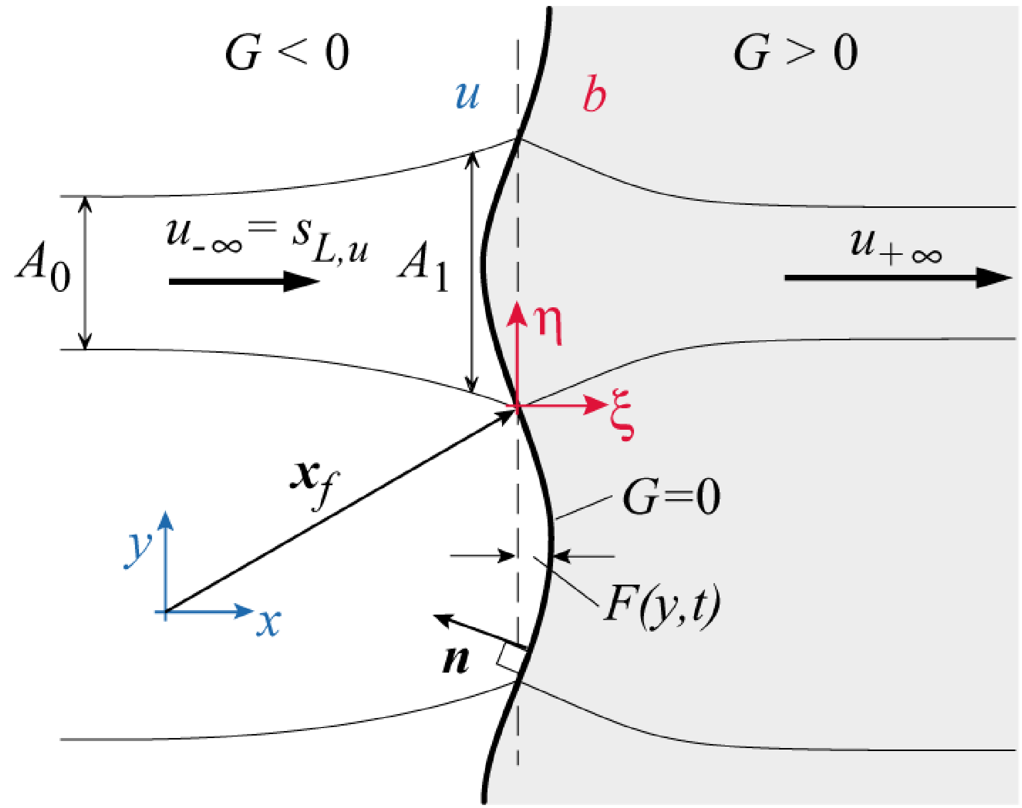
Flame Instabilities: Hydrodynamic Instability

- Illustration of the hydro-dynamic instability of a slightly undulated flame

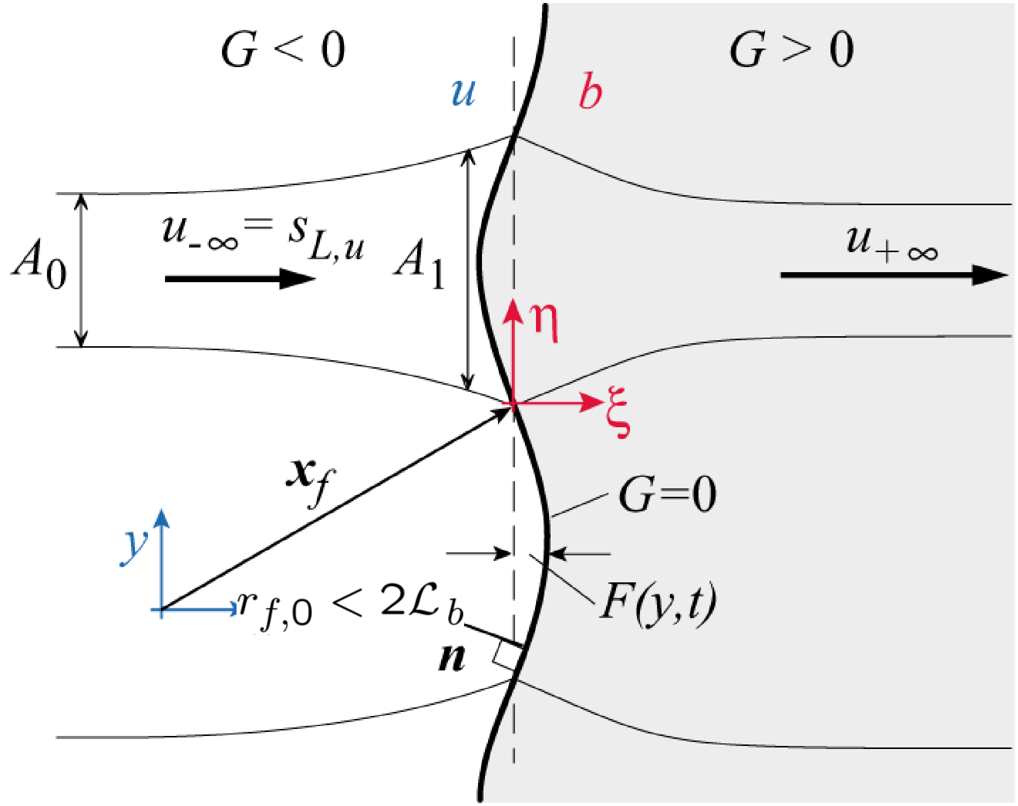


- Gas expansion in the flame front leads to a deflection of a stream line that enters the front at an angle
- A stream tube with cross-sectional area A_0 and upstream flow velocity $u_{-\infty}$ widens due to flow divergence ahead of the flame

Flame Instabilities: Hydrodynamic Instability

- Expansion at the front induces a flow component normal to the flame contour
- As the stream lines cross the front they are deflected
- At large distances from front, stream lines are parallel again, but downstream velocity is

$$u_{+\infty} = (\rho_u/\rho_b)\, u_{-\infty}$$



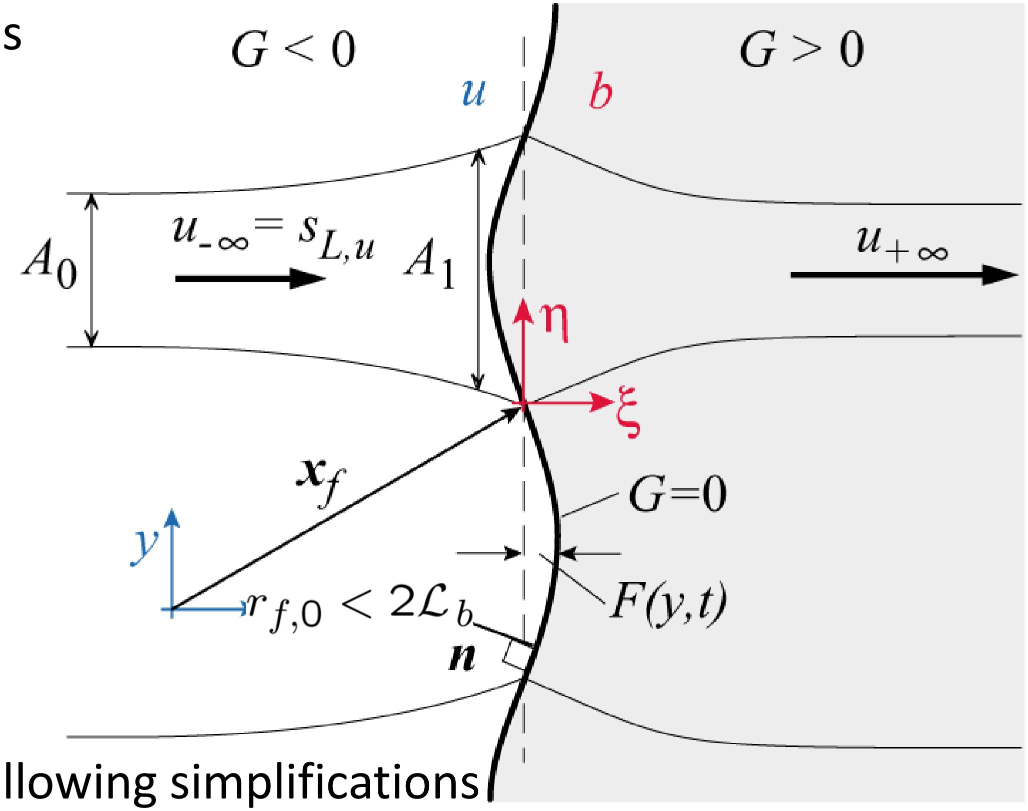
- At a cross section A_1, where density is still equal to ρ_u , by continuity flow velocity becomes

$$u_1 = \frac{A}{A_1} u_{-\infty} \leq u_{-\infty}.$$

Flame Instabilities: Hydrodynamic Instability

- The unperturbed flame propagates with $u_{-\infty} = s_{L,u}$ normal to itself

- Burning velocity is larger than u_1, flame propagates upstream and thereby enhances the initial perturbation



- Analysis can be performed with following simplifications
  - Viscosity, gravity and compressibility in the burnt and unburnt gas are neglected
  - Density is discontinuous at the flame front
  - The influence of the flame curvature on the burning velocity is retained, flame stretch due to flow divergence is neglected

Flame Instabilities: Hydrodynamic Instability

- Analysis results in dispersion relation

$$\sigma = \frac{s_{L0}^{-}k}{1+r}\left\{\sqrt{1+k^2\mathcal{L}^2 - \frac{2k\mathcal{L}}{r} + \frac{1-r^2}{r}} - (1+k\mathcal{L})\right\}$$

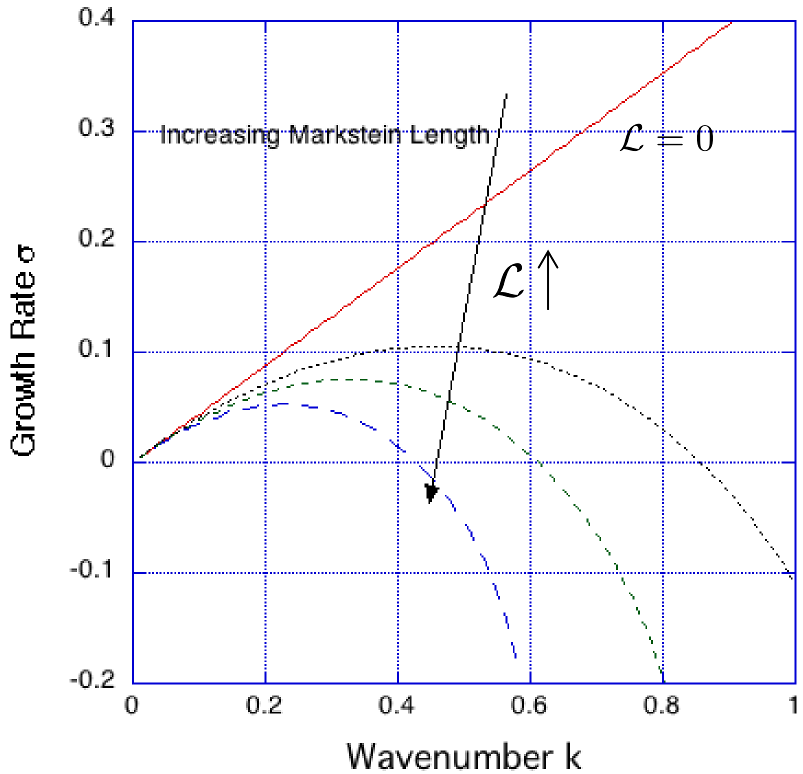
where σ is the non-dimensional growth rate of the perturbation

$$\sigma = \frac{1}{f}\frac{df}{dt} = \frac{d\ln f}{dt}$$

r is density ratio and *k* the wave number

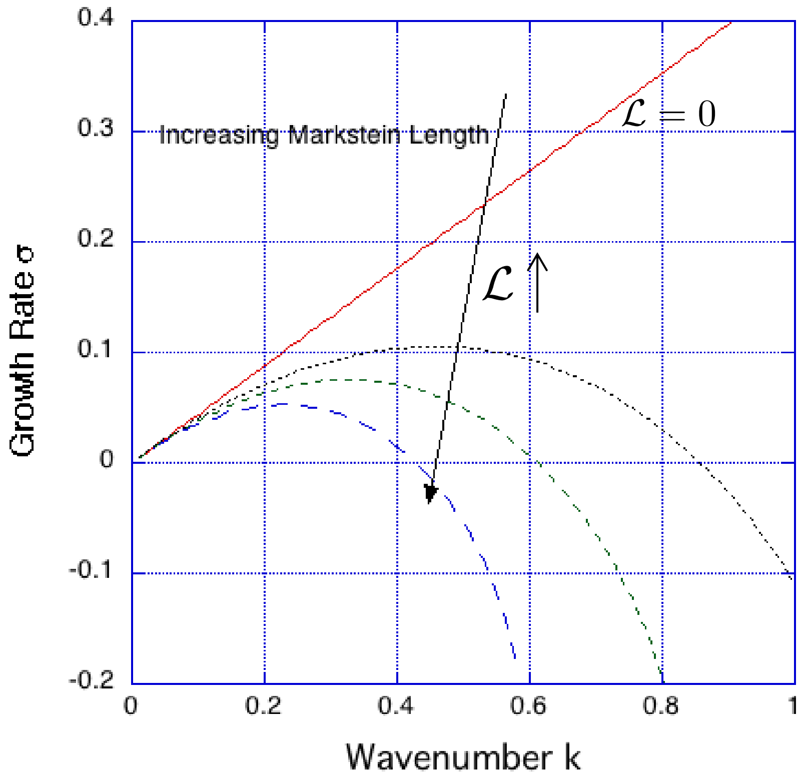
- Perturbation grows exponentially in time only for a certain wavenumber range $0 < k < k^*$ with

$$k^* = (r-1)/(2\mathcal{L}).$$



Flame Instabilities: Hydrodynamic Instability

- Without influence of curvature ($\mathcal{L} = 0$), flame is unconditionally unstable
- For perturbations at wave numbers $k > k^*$, a planar flame of infinitively small thickness, described as a discontinuity in density, velocity and pressure is unconditionally stable
  - Influence of front curvature on burning velocity
- As one would expect on the basis of simple thermal theories of flame propagation, burning velocity increases when flame front is concave and decreases when it is convex towards unburnt gas, so that initial perturbations become smoother



*Details of the Analysis for Hydrodynamic Instability

- The burning velocity is given by

$$s_L = s_L^0 (1 + \kappa \mathcal{L})$$

- Reference values for length, time, density, pressure:

$$\ell_F \,, \quad \ell_F / s_{L,u} \,, \quad \rho_u \,, \quad \rho_u s_{L,u}^2$$

- Introduce the density rate:

$$r = \rho_b / \rho_u < 1$$

- Dimensionless variables:

$$u^* = u/s_{L,u}, \quad v^* = v/s_{L,u}, \quad p^* = \frac{p}{\rho_u s_{L,u}^2},$$

$$x^* = x/\ell_F, \quad y^* = y/\ell_F, \quad t^* = \frac{t}{\ell_F / s_{L,u}}.$$

*Details of the Analysis for Hydrodynamic Instability

- The non-dimensional governing equations are then (with the asterisks removed)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

where $\rho_u = 1$ and $\rho = r$ in the unburnt and burnt mixture respectively.

- If *G* is a measure of the distance to the flame front, the *G*-field is described by:

$$G = x - F(y, t)$$

*Details of the Analysis for Hydrodynamic Instability

- With equations

$$\boldsymbol{n} = -\frac{\nabla G}{|\nabla G|}, \qquad \frac{\partial G}{\partial t} - |\nabla G| \, \boldsymbol{n} \cdot \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{G=G_0} = 0$$

the normal vector ***n*** and the normal propagation velocity then are

$$\boldsymbol{n} = \left(-1, \, \frac{\partial F}{\partial y}\right) \Big/ \sqrt{1 + \left(\frac{\partial F}{\partial y}\right)^2}, \quad \boldsymbol{n} \cdot \left.\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|_{G=G_0} = \frac{\partial F}{\partial t} \Big/ \sqrt{1 + \left(\frac{\partial F}{\partial y}\right)^2}$$

*Details of the Analysis for Hydrodynamic Instability

- Due to the discontinuity in density at the flame front, the Euler equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

  are only valid on either side of the front, but do not hold across it.

- Therefore jump conditions for mass and momentum conservation across the discontinuity are introduced [Williams85,p. 16]:

$$(r-1)\,\boldsymbol{n}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\bigg|_{G=G_0} = \boldsymbol{n}\cdot(r\boldsymbol{v}_+ - \boldsymbol{v}_-)$$
$$(r\boldsymbol{v}_+ - \boldsymbol{v}_-)\,\boldsymbol{n}\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\bigg|_{G=G_0} = \boldsymbol{n}\cdot\Big(r\boldsymbol{v}_+\boldsymbol{v}_+ - \boldsymbol{v}_-\boldsymbol{v}_- - (p_+ - p_-)\boldsymbol{I}\Big)$$

- The subscripts + and - refer to the burnt and the unburnt gas and denote the properties immediately downstream and upstream of the flame front.

*Details of the Analysis for Hydrodynamic Instability

- In terms of the *u* and v components the jump conditions read

$$
\begin{aligned}
(r-1)\frac{\partial F}{\partial t} &= ru_+ - u_- - \frac{\partial F}{\partial y}(rv_+ - v_-) \\
(ru_+ - u_-)\frac{\partial F}{\partial t} &= ru_+(u_+ - \frac{\partial F}{\partial y}v_+) - u_-(u_- - \frac{\partial F}{\partial y}v_-) + p_+ - p_- \\
(rv_+ - v_-)\frac{\partial F}{\partial t} &= rv_+(u_+ - \frac{\partial F}{\partial y}v_+) - v_-(u_- - \frac{\partial F}{\partial y}v_-) - \frac{\partial F}{\partial y}(p_+ - p_-).
\end{aligned}
$$

- Under the assumption of small perturbations of the front, with e << 1 the unknowns are expanded as

$$u = U + \epsilon u, \qquad v = \epsilon v$$

$$p = P + \epsilon p, \qquad F = \epsilon f,$$

*Details of the Analysis for Hydrodynamic Instability

- Jump conditions to leading order

$$U_- = 1, \quad P_- = 0$$

$$U_+ = \frac{1}{r}, \quad P_+ = \frac{r-1}{r},$$

and to first order

$$\begin{aligned} (r-1)\frac{\partial f}{\partial \tau} &= ru_+ - u_- \\ 0 &= 2(u_+ - u_-) + p_+ - p_- \\ 0 &= v_+ - v_- + \frac{1-r}{r}\frac{\partial f}{\partial \eta}, \end{aligned}$$

where the leading order mass flux has been set equal to one:

$$\dot{m} = rU_+ = U_- = 1$$

*Details of the Analysis for Hydrodynamic Instability

- With the coordinate transformation $x = \xi + F(\eta, \tau), \quad y = \eta, \quad t = \tau$ we fix the discontinuity at x = 0.

- To first order the equations for the perturbed quantities on both sides of the flame front now read

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0$$

$$\frac{\partial u}{\partial \tau} + U\frac{\partial u}{\partial \xi} + \frac{1}{\rho}\frac{\partial p}{\partial \xi} = 0$$

$$\frac{\partial v}{\partial \tau} + U\frac{\partial u}{\partial \xi} + \frac{1}{\rho}\frac{\partial p}{\partial \eta} = 0,$$

where $\rho = 1$ for $\xi < 0$ (unburnt gas) and $\rho = r$ for $\xi > 0$ (burnt gas) is to be used.

- In case of instability perturbations which are initially periodic in the h-direction and vanish for $x \rightarrow \pm\infty$ would increase with time.

*Details of the Analysis for Hydrodynamic Instability

- Since the system is linear, the solution may be written as

$$\boldsymbol{w} = \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \boldsymbol{w}_0 \exp(\alpha\xi) \exp(\sigma\tau - \mathrm{i}\mathrm{k}\eta),$$

where σ is the non-dimensional growth rate, κ the non-dimensional wave number and i the imaginary unit.

- Introducing this into the first order equations the linear system may be written as

$$\boldsymbol{A} \cdot \boldsymbol{w} = \boldsymbol{0}$$

- The matrix ***A*** is given by

$$\boldsymbol{A} = \begin{pmatrix} \alpha & -\mathrm{ik} & 0 \\ \sigma + \alpha U & 0 & \alpha/\rho \\ 0 & \sigma + \alpha U & -\mathrm{ik}/\rho \end{pmatrix}.$$

*Details of the Analysis for Hydrodynamic Instability

- The eigenvalues of ***A*** are obtained by setting det(***A***) = 0.

- This leads to the characteristic equation

$$\det(\boldsymbol{A}) = \frac{1}{\rho}\left(k^2 - \alpha^2\right)\left(\sigma + \alpha U\right) = 0.$$

- Here again $U = 1/r$, $\rho = r$ for $\xi > 0$ and $U = 1$, $\rho = 1$ for $\xi < 0$.

- There are three solutions to the characteristic equation for the eigenvalues α_j, $j = 1,2,3$.

- Positive values of a_j satisfy the upstream ($\xi < 0$) and negative values the downstream ($\xi > 0$) boundary conditions of the Euler equations.

*Details of the Analysis for Hydrodynamic Instability

- Therefore

$$\xi > 0: \quad \alpha_1 = -r\sigma, \quad \alpha_2 = -k$$

$$\xi < 0: \quad \alpha_2 = -k.$$

- Introducing the eigenvalues into $\boldsymbol{A} \cdot \boldsymbol{w} = \boldsymbol{0}$ again, the corresponding eigenvectors $\boldsymbol{w}_{0,j}$, j = 1,2,3 are calculated to

$$j = 1: \qquad w_{0,1} = \left(1, \quad \mathrm{i}\frac{\mathrm{r}\sigma}{\mathrm{k}}, \quad 0 \right)$$

$$j = 2: \qquad w_{0,2} = \left(1, \quad \mathrm{i}, \quad -1 + \frac{\mathrm{r}\sigma}{\mathrm{k}} \right)$$

$$j = 3: \qquad w_{0,3} = \left(1, \quad -\mathrm{i}, \quad -1 - \frac{\sigma}{\mathrm{k}} \right)$$

*Details of the Analysis for Hydrodynamic Instability

- In terms of the original unknowns *u*, *v* and the solution is now

$$\xi > 0: \quad \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \left\{ a \begin{pmatrix} 1 \\ \mathrm{i}\dfrac{r\sigma}{\mathrm{k}} \\ 0 \end{pmatrix} \exp(-r\sigma\xi) + b \begin{pmatrix} 1 \\ \mathrm{i} \\ -1 + \dfrac{r\sigma}{k} \end{pmatrix} \exp(-k\xi) \right\} \exp(\sigma\tau - \mathrm{i}\mathrm{k}\eta)$$

$$\xi < 0: \quad \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \qquad c \begin{pmatrix} 1 \\ -\mathrm{i} \\ -1 - \dfrac{\sigma}{k} \end{pmatrix} \exp(k\xi + \sigma\tau - \mathrm{i}\mathrm{k}\eta).$$

- For the perturbation $f(\eta, \tau)$ the form

$$f = \tilde{f} \exp(\sigma\tau - \mathrm{i}k\eta)$$

will be introduced.

*Details of the Analysis for Hydrodynamic Instability

- Inserting $$\kappa = -\frac{\nabla^2 G + \boldsymbol{n}\cdot\nabla(\boldsymbol{n}\cdot\nabla G)}{|\nabla G|}\,, \qquad G = x - F(y,t)\,,$$

  and

$$u = U + \epsilon u, \qquad v = \epsilon v$$

$$p = P + \epsilon p, \qquad F = \epsilon f,$$

into the non-dimensional *G*-equation

$$\left(\frac{\partial G}{\partial t} + u\frac{\partial G}{\partial x} + v\frac{\partial G}{\partial y}\right) = \sqrt{\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2}\,(1+\kappa\mathcal{L})$$

satisfies to leading order with

$$u = U + \epsilon u, \qquad v = \epsilon v$$

$$p = P + \epsilon p, \qquad F = \epsilon f,$$

and $x = 0_-$, $x = 0_+$ respectively.

*Details of the Analysis for Hydrodynamic Instability

- This leads to first order to

$$u_- = \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial \eta^2}\mathcal{L}$$

$$u_+ = \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial \eta^2}\frac{\mathcal{L}}{r}.$$

- With

$$f = \tilde{f}\,\mathrm{exp}(\sigma\tau - \mathrm{i}k\eta)$$

the jump conditions

$$\begin{aligned}(r-1)\,\frac{\partial f}{\partial \tau} &= r u_+ - u_- \\ 0 &= 2(u_+ - u_-) + p_+ - p_- \\ 0 &= v_+ - v_- + \frac{1-r}{r}\frac{\partial f}{\partial \eta},\end{aligned}$$

- can be written as

$$\begin{aligned}(r-1)\,\sigma\tilde{f} &= r\,(a+b) - c \\ 0 &= 2a + b\,(1 + r\frac{\sigma}{k}) + c\,(\frac{\sigma}{k} - 1) \\ \frac{1-r}{r}k\tilde{f} &= a\frac{r\sigma}{k} + b + c\end{aligned}$$

*Details of the Analysis for Hydrodynamic Instability

- The system

$$u_- = \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial \eta^2}\mathcal{L}$$

$$u_+ = \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial \eta^2}\frac{\mathcal{L}}{r}$$

then reads

$$c = \tilde{f}\,(\sigma + k^2\mathcal{L})$$

$$a + b = \tilde{f}\,(\sigma + \frac{k^2\mathcal{L}}{r})$$

*Details of the Analysis for Hydrodynamic Instability

- Since equation

$$(r-1)\,\sigma\tilde{f} \;=\; r\,(a+b)-c$$

is linear dependent from equations

$$c = \tilde{f}\,(\sigma + k^2\mathcal{L})$$

$$a+b = \tilde{f}\,(\sigma + \frac{k^2\mathcal{L}}{r})$$

it is dropped and the equations

$$0 \;=\; 2a + b\,(1 + r\frac{\sigma}{k}) + c\,(\frac{\sigma}{k} - 1)$$

$$\frac{1-r}{r}k\tilde{f} \;=\; a\frac{\sigma}{k} + b + c$$

nd

$$c = \tilde{f}\,(\sigma + k^2\mathcal{L})$$

$$a+b = \tilde{f}\,(\sigma + \frac{k^2\mathcal{L}}{r})$$

remain for the determination of a, b, c and s(*k*).

*Details of the Analysis for Hydrodynamic Instability

- Dividing all equations by $k\,\tilde{f}$ one obtains four equations for

$$\widehat{a} = a/(k\tilde{f})\,, \quad \widehat{b} = b/(k\tilde{f})\,, \quad \widehat{c} = c/(k\tilde{f})\,, \quad \varphi = \sigma/k$$

- The elimination of the first three unknown yields the equation

$$\varphi^2(1+r) + 2\varphi(1+k\mathcal{L}) + \frac{2k\mathcal{L}}{r} + \frac{r-1}{r} = 0$$

- The solution may be written in terms of dimensional quantities as

$$\sigma = \frac{s_{L0}^{-}k}{1+r}\left\{\sqrt{1+k^2\mathcal{L}^2 - \frac{2k\mathcal{L}}{r} + \frac{1-r^2}{r}} - (1+k\mathcal{L})\right\}$$

- Here only the positive root has been taken, since it refers to possible solutions with exponential growing amplitudes.

*Details of the Analysis for Hydrodynamic Instability

The relation

$$\sigma = \frac{s_{L0}^{-} k}{1+r}\left\{\sqrt{1+k^2\mathcal{L}^2-\frac{2k\mathcal{L}}{r}+\frac{1-r^2}{r}}-(1+k\mathcal{L})\right\}$$

is the dispersion relation which shows that the perturbation *f* grows exponentially in time only for a certain wavenumber range $0 < k < k^*$.

Here k^* is the wave number of which $\varphi = 0$ in

$$\varphi^2(1+r)+2\varphi(1+k\mathcal{L})+\frac{2k\mathcal{L}}{r}+\frac{r-1}{r}=0$$

which leads to

$$k^* = (r-1)/(2\mathcal{L}).$$

*Exercise

- Under the assumption of a constant burning velocity $s_L = s_{L0}$ the linear stability analysis leads to the following dispersion relation

$$\sigma = \frac{s_{L0}^{-}k}{1+r}\left\{\sqrt{1+\frac{1-r^2}{r}}-1\right\}.$$

- Validate this expression by inserting $\mathcal{L} = 0$

$$\sigma = \frac{s_{L0}^{-}k}{1+r}\left\{\sqrt{1+k^2\mathcal{L}^2-\frac{2k\mathcal{L}}{r}+\frac{1-r^2}{r}}-(1+k\mathcal{L})\right\}$$

- What is the physical meaning of this result?
- What effect has the front curvature on the flame front stability?

*Exercise

Solution

- The dispersion relation for constant burning velocity $s_L = s_{L0}$,

$$\sigma = \frac{s_{\bar{L}0} k}{1+r} \left\{ \sqrt{1 + \frac{1-r^2}{r}} - 1 \right\}.$$

shows that the perturbation *F* grows exponentially in time for all wave numbers.

- The growth s is proportional to the wave number *k* and always positive since the density rate r is less than unity.

- This means that a plane flame front with constant burning velocity is unstable to any perturbation.

*Exercise

- The front curvature has a stabilizing effect on the flame front stability.

- As it is shown in the last section, the linear stability analysis for a burning velocity with the curvature effect retained leads to instability of the front only for the wave number range

$$0 < k < k^* = (r - 1)/(2\mathcal{L}),$$

whereas the front is stable to all perturbations with k > k^*.

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